Linearized Metagravity from Conformal Deformations

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Abstract

The principle of *relative locality* is a generalization of the principle of relativity in which even locality – the coincidence of events – is no longer absolute, and each observer has a different notion of spacetime. This is achieved by allowing (energy-)momentum space to be curved independently of spacetime. Phase space is the only truly invariant structure, that is, independent of the observer. This principle has recently been implemented in *metastring theory*, a new formulation of string theory in which the strings propagate on the entire phase space. In addition to the symplectic form ω, the metastring phase space possesses a polarization metric η, which specifies how phase space is decomposed into spacetime and momentum space, and a generalized metric H, which encodes the independent curvatures of these spaces. Our aim in this essay is to make the first step towards understanding this so-called *metageometry*, postulated to uniquely define a generalization of Einstein gravity which we refer to as *metagravity*. This is accomplished by conjuring some "string magic". If we ask the strings of string theory what kind of background spacetime they are willing to propagate on, their reply, compelled by mathematical consistency alone, is that the spacetime metric must satisfy Einstein's equations of general relativity. Deriving the metagravity equations is thus simply a matter of posing the same question in metastring theory. More precisely, we perturb the background fields and require that the resulting worldsheet theory remains a conformal field theory. Using this method of *conformal deformations*, we find linearized equations of motion for the perturbations.

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1 Introduction

1.1 Relative Locality

In Einstein's theory of relativity, there is no notion of absolute space or absolute time. Instead, both space and time are relative; two observers in different reference frames will generally not agree on measurements of the length and duration of an event. Space and time may be said to emerge from the more fundamental notion of *spacetime*, which is absolute and does not depend on the observer. Locality is also absolute; that is, any two observers will agree that two events coincide.

The principle of *relative locality* [1] is a generalization of the relativity principle. In relative locality, spacetime is no longer absolute, nor is locality. The absolute, fundamental structure is phase space, and each observer's individual notion of spacetime emerges from it. Spacetime becomes energy and momentum dependent. Two different observers will in general not agree, not only on length or durations, but also on the coincidence of events. Absolute locality is thus replaced by relative locality.

The energy and momentum dependence of spacetime in relative locality comes from the curvature of momentum space. Adding momenta and energies is now a nonlinear operation, similar to addition of velocities in relativity. In addition to the invariant velocity c, we now have an invariant energy/momentum scale ε, to be determined experimentally[1]. The effects of relative locality should only becomes noticeable at energy/momentum scales close to the invariant scale ε.

1.2 Metastring Theory

Metastring theory is a reformulation of string theory recently introduced by Freidel, Leigh and Minic [2] (see also [3, 4]). This reformulation generalizes string theory, relaxes some assumptions which are usually taken for granted such as locality, and introduces several novel concepts such as *modular spacetime*.

Most importantly for our particular context, it incorporates a notion of relative locality. This is done by replacing the target space of the worldsheet nonlinear sigma model, usually taken to be spacetime, with the entire phase space, in a way that is compatible with standard bosonic string theory. The strings of metastring theory – or metastrings – thus propagate in phase space. This phase space may be curved, and the curvature generally depends on both spacetime and momentum-space coordinates.

The geometry of phase space in metastring theory, which we shall refer to as the *metageometry*, includes three fundamental geometric structures. Since the phase space $\mathcal{P}$ is $2D$-dimensional, where D is the dimension of spacetime, these structures are $2D$-dimensional. The first structure is a symplectic form, ω, which is the usual symplectic form associated with any phase space, and in metastring theory it is allowed to be dynamical. As we will see, both spacetime L and momentum space $\tilde{L}$ are Lagrangian submanifolds of $\mathcal{P}$, meaning manifolds of maximum dimension on which ω vanishes.

[1] This may or may not be the Planck mass.

The second structure is the *polarization metric* (P-metric) η. It has the property that both spacetime L and momentum space $\tilde{L}$ are null subspaces with respect to η. Thus, when η is allowed to be dynamical, the definition of spacetime and momentum space also changes. The choice of L and $\tilde{L}$ as Lagrangian submanifolds of $\mathcal{P}$ is called a polarization. These two manifolds are transverse, making up a bilagrangian structure on $\mathcal{P}$ such that $T\mathcal{P} = TL \oplus T\tilde{L}$ and $TL \cap T\tilde{L} = \{0\}$. They may be defined by $L \equiv \ker(\eta + \omega)$ and $\tilde{L} \equiv \ker(\eta - \omega)$.

Finally, the third structure is the generalized metric or *quantum metric* (Q-metric) H. It is a generalization of the standard spacetime metric, encoding the curvature of both spacetime and momentum space, which in general are allowed to be completely independent. When restricted to the spacetime submanifold L, it reduces to the usual spacetime metric.

Using these three fundamental structures we may construct additional derived structures. The most important of these are J, the *chiral structure*, and the *chiral projectors* $P_{\pm}$. We will discuss them at length below.

Two other structures of interest on the metageometry are a complex structure I and a real structure K. Together, I, J and K possess a para-quaternionic structure, $-I^2 = J^2 = K^2 = 1$ and $IJK = -1$. The metageometry described by them is referred to as *Born geometry*; it unifies the complex geometry of quantum theory, the symplectic geometry of Hamiltonian dynamics and the real geometry of general relativity. Unfortunately, we will not have the opportunity to discuss Born geometry at more depth in this work.

1.3 Conformal Deformations and Metagravity

In string theory, the strings propagate on a fixed background spacetime given by three massless fields: the spacetime metric $g_{\mu\nu}$, the 2-form $B_{\mu\nu}$ and the scalar dilaton Φ. Upon quantization, these fields are seen to coincide with excited states of closed strings. The standard Polyakov action of bosonic string theory defines a 2-dimensional conformal field theory of D massless scalar fields X^μ, which are none other than the spacetime coordinates themselves. This is known as a nonlinear sigma model, and spacetime is referred to as the target space. Different spacetime backgrounds $g_{\mu\nu}$, $B_{\mu\nu}$ and Φ then correspond to a different choice of conformal field theory for the string worldsheet.

If we perturb the background fields, we change, or *deform*, the worldsheet theory. An important question is, therefore, what kinds of perturbations are allowed such that the deformed worldsheet theory is still a conformal field theory, and thus still describes a string? We will investigate this question only for perturbations of the spacetime metric and assuming the B-field and dilaton vanish, using the method of *conformal deformations*, originally developed for string field theory [5, 6, 7, 8]. We shall discover that the allowed perturbations are exactly those which satisfy the linearized Einstein field equations. In other words, the string tell us it wants to propagate only on a background that obeys the (vacuum) Einstein equations. In some sense a string, presumably the smallest thing in existence, controls the behavior of the entire universe!

It is therefore natural to attempt to apply this method to the phase space metageometry described by metastring theory. Indeed, we shall see that metastrings will only propagate on a metageometry obeying a specific set of equations, generalizing Einstein gravity to a new theory of *metagravity*[2].

1.4 Notation and Conventions

We will mostly follow the notation and conventions of [9]. For brevity we take $\alpha' \equiv 2$ everywhere. D is always the number of spacetime dimensions.

The meaning of indices is taken to be as follows:

[2]Gravity is the dynamical curvature of spacetime. However, in relative locality, spacetime itself is a relative notion. Thus, before we are able to describe gravity on spacetime, we must first refer to the metageometry in order to determine what spacetime is. The restriction of the dynamics to this particular submanifold would then give us the usual notion of gravity. The dynamical curvature of phase space is thus, in a sense, "gravity of gravity", or *metagravity*.

- Lowercase Latin indices from the beginning of the alphabet $a, b, c, \ldots \in \{0, 1\}$ or $\in \{1, 2\}$: Lorentzian or Euclidean worldsheet coordinates, respectively.
- Lowercase Latin indices from the middle of the alphabet $\ldots, l, m, n, \ldots \in \{0, 1\}$: internal frame field coordinates on a Lorentzian worldsheet.
- Lowercase Latin indices from the end of the alphabet $\ldots, x, y, z \in \{0, 1\}$: complex/chiral worldsheet coordinates.
- Uppercase Latin indices from the beginning of the alphabet $A, B, C, \ldots \in \{1, \ldots, 2D\}$: "unified" phase space coordinates.
- Uppercase Latin indices from the end of the alphabet $\ldots, X, Y, Z \in \{+, -\}$: phase space chiral projection components.
- Lowercase Greek indices $\alpha, \beta, \ldots, \mu, \nu, \ldots \in \{1, \ldots, D\}$: spacetime or momentum-space coordinates.

If an object has both phase space indices $A, B, \ldots$ and chiral projection indices $X, Y, \ldots$ then they will be separated by parentheses and the projection indices will always be the inner indices, for example $(V_{XY})_{AB}$.

Vectors will always have upper indices $(X^\mu, \mathbb{X}^A, \ldots)$ while covectors will always have lower indices $(P_\mu, \mathbb{P}_A, \ldots)$.

We use the following metrics:

- H_{AB} is the $2D$-dimensional phase space Q-metric,
- η_{AB} is the $2D$-dimensional phase space P-metric,
- $h_{\mu\nu}$ is the D-dimensional spacetime Minkowski metric,
- γ_{ab} is the 2-dimensional worldsheet Minkowski metric.

The coordinates on a Lorentzian worldsheet are τ, σ, and we define $\sigma^0 \equiv \tau$, $\sigma^1 \equiv \sigma$. The coordinates on a Euclidean worldsheet are σ^1, σ^2.

Holomorphic fields on the string worldsheet are also referred to as "left-moving", while antiholomorphic fields are "right-moving". Projections on the $+$ $(-)$ eigenspace of the chiral structure J on the target space correspond to holomorphic (antiholomorphic) fields on the worldsheet.

2 Conformal Deformations in Standard String Theory

We first discuss deformations of the worldsheet conformal field theory in standard string theory, and show how they automatically give rise to the linearized Einstein equations.

2.1 Deformations of Conformal Field Theories

For our purposes, a 2-dimensional *conformal field theory* parametrized by complex coordinates $z, \bar{z}$ is defined by the existence of a traceless energy-momentum tensor with a holomorphic (left-moving) component T and an antiholomorphic (right-moving) component $\bar{T}$ which satisfy the *operator product expansions* (OPEs):

$$
\begin{aligned}
T(z)\,T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \\
\bar{T}(\bar{z})\,\bar{T}(\bar{w}) &\sim \frac{\bar{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{w})}{\bar{z}-\bar{w}}, \\
T(z)\,\bar{T}(\bar{w}) &\sim 0,
\end{aligned}
\tag{2.1}
$$

where the notation $\sim$ means "up to nonsingular terms as $z, \bar{z}$ approach $w, \bar{w}$". Here c and $\bar{c}$ are the left-moving and right-moving *central charges*, respectively, and might in general be different.

A field $\Phi(z, \bar{z})$ is a *primary field* of *weight* $(h, \bar{h})$ if and only if it has the following OPEs with T and $\bar{T}$:

$$T(z)\Phi(w, \bar{w}) \sim \frac{h\Phi(w, \bar{w})}{(z-w)^2} + \frac{\partial\Phi(w, \bar{w})}{z-w}, \qquad (2.2)$$

$$\bar{T}(\bar{z})\Phi(w, \bar{w}) \sim \frac{\bar{h}\Phi(w, \bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\Phi(w, \bar{w})}{\bar{z}-\bar{w}}.$$

We see that $T(z)$ is a field of weight $(2,0)$ and $\bar{T}(\bar{z})$ is a field of weight $(0,2)$, but neither of them are primary fields due to the $1/z^4$ term in the OPE. Alternatively, a field $\Phi(z, \bar{z})$ is a primary field of weight $(h, \bar{h})$ if it transforms under a conformal transformation $z \mapsto w(z)$ and $\bar{z} \mapsto \bar{w}(\bar{z})$, where $w, \bar{w}$ are arbitrary (anti)holomorphic functions, as

$$\Phi(z, \bar{z}) \mapsto \Phi'(w, \bar{w}) \equiv \left(\frac{\partial w}{\partial z}\right)^{-h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}} \Phi(z, \bar{z}). \qquad (2.3)$$

Let us now define an infinitesimal perturbation of the components of the energy-momentum tensor,

$$T(z) \mapsto T(z) + \delta T(z, \bar{z}), \qquad \bar{T}(\bar{z}) \mapsto \bar{T}(\bar{z}) + \delta\bar{T}(z, \bar{z}), \qquad (2.4)$$

where δT and $\delta\bar{T}$ are in general independent. By perturbing the energy-momentum tensor, we are *deforming* the conformal field theory into a new theory, defined by the new energy-momentum tensor components $T + \delta T$ and $\bar{T} + \delta\bar{T}$. We will only be interested in *conformal deformations*, for which the new theory is also a conformal field theory. In other words, conformal deformations are those that preserve the conformal symmetry.

Achiral canonical deformations (or just *canonical deformations*) are deformations where $\delta T = \delta\bar{T} = \Phi$ and Φ is a $(1,1)$ primary field. Such deformations are automatically conformal. They naturally appear in string theory, where $(1,1)$ primary fields are the vertex operators corresponding to physical states of the string.

Chiral canonical deformations are a generalization of achiral canonical deformations where δT and $\delta\bar{T}$ are two independent $(1,1)$ primary fields; that is, in general $\delta T \neq \delta\bar{T}$. As we shall see, chiral canonical deformations naturally appear in metastring theory.

More general conformal deformations, by $(1,1)$ primary fields plus additional boundary deformations, may be used to study the symmetries of the theory [6]. They will be thoroughly investigated in future work.

2.2 The Polyakov Action

We begin by introducing the worldsheet CFT of bosonic string theory and deriving some basic results. These derivations are given here mainly to enable comparison with the subsequent derivation of similar results in metastring theory. The educated reader should feel free to skip ahead to section 2.6.

The Polyakov action for a bosonic string propagating in flat spacetime is

$$S_{\mathrm{P}}[X] \equiv \frac{1}{8\pi} \int_{\Sigma} h_{\mu\nu} (\star \mathrm{d}X^{\mu} \wedge \mathrm{d}X^{\nu}), \qquad (2.5)$$

where Σ is the 2-dimensional string worldsheet with local coordinates (σ, τ) and Lorentzian metric, $\star$ is the Hodge dual on the worldsheet such that $\star\mathrm{d}\tau = \mathrm{d}\sigma$ and $\star^2 = 1$, d is the exterior derivative on the worldsheet and X^{μ} are local coordinates on a D-dimensional[3] target space M with Minkowski

[3] In order to avoid a Weyl anomaly in the quantum theory, the central charge c of the worldsheet CFT must vanish. As it turns out, canceling the unphysical gauge degrees of freedom in the path integrals requires the introduction of Faddeev-Popov ghost fields, which contribute -26 to the central charge. Since the fields X^{μ} each contribute $+1$ to the central charge, we must have 26 of them, which means that the target space must have $D = 26$ dimensions. This fact will not, however, be of particular significance to us in this work.

metric $h_{\mu\nu}$, which we interpret as a (flat) spacetime. Note that the numerical factor is $1/4\pi\alpha' = 1/8\pi$, as we are taking $\alpha' \equiv 2$.

This action may be put into a more familiar form by writing it in local coordinates, with Lorentzian worldsheet metric γ_{ab}. This gives

$$S_{\mathrm{P,L}}[X] = -\frac{1}{8\pi}\int \mathrm{d}^2\sigma\,\sqrt{-\gamma}\,\gamma^{ab}\partial_a X^\mu \partial_b X^\nu h_{\mu\nu}, \tag{2.6}$$

where $\mathrm{d}^2\sigma \equiv \mathrm{d}\sigma \wedge \mathrm{d}\tau$ and $\gamma \equiv \det \gamma_{ab}$. The subscript L denotes Lorentzian signature. We define the expectation value of an arbitrary functional $F[X]$ with respect to the Lorentzian Polyakov action using a path integral as follows:

$$\langle F \rangle \equiv \int \mathcal{D}X\, \mathrm{e}^{\mathrm{i}\, S_{\mathrm{P,L}}[X]}\, F[X]. \tag{2.7}$$

We switch from the Lorentzian coordinates (σ, τ) to Euclidean coordinates (σ^1, σ^2) by performing a Wick rotation: $\sigma^1 \mapsto \sigma$ and $\sigma^2 \mapsto -\mathrm{i}\,\tau$. The action receives a factor of i. By convention we would like the exponent in the Euclidean path integral to have a negative sign:

$$\langle F \rangle \equiv \int \mathcal{D}X\, \mathrm{e}^{-S_{\mathrm{P,E}}[X]}\, F[X], \tag{2.8}$$

where the subscript E denotes Euclidean signature. This prompts us to define the Euclidean Polyakov action with the opposite sign compared to the Lorentzian one:

$$S_{\mathrm{P,E}}[X] = \frac{1}{8\pi}\int \mathrm{d}^2\sigma\,\sqrt{\gamma}\,\gamma^{ab}\partial_a X^\mu \partial_b X^\nu h_{\mu\nu}, \tag{2.9}$$

where now γ_{ab} is a Euclidean worldsheet metric.

Finally, we switch from the Euclidean coordinates (σ^1, σ^2) to complex coordinates $(z, \bar{z})$:

$$z \equiv \sigma^1 + \mathrm{i}\,\sigma^2, \qquad \bar{z} \equiv \sigma^1 - \mathrm{i}\,\sigma^2. \tag{2.10}$$

We define the holomorphic and antiholomorphic partial derivatives as

$$\partial \equiv \partial_z \equiv \frac{1}{2}(\partial_1 - \mathrm{i}\,\partial_2), \qquad \partial_{\bar{z}} \equiv \bar{\partial} \equiv \frac{1}{2}(\partial_1 + \mathrm{i}\,\partial_2), \tag{2.11}$$

respectively. This gives the desired result

$$\partial z = \bar{\partial}\bar{z} = 1, \qquad \partial\bar{z} = \bar{\partial} z = 0. \tag{2.12}$$

We can also invert these relations to obtain the Euclidean derivatives in terms of the complex derivatives:

$$\partial_1 = \partial + \bar{\partial}, \qquad \partial_2 = \mathrm{i}\left(\partial - \bar{\partial}\right). \tag{2.13}$$

Hence, after Wick-rotating, the Lorentzian derivatives are replaced with the complex derivatives as follows:

$$\partial_\sigma \mapsto \partial + \bar{\partial}, \qquad \partial_\tau \mapsto \partial - \bar{\partial}. \tag{2.14}$$

The flat Euclidean metric becomes, in complex coordinates, the off-diagonal metric

$$\gamma_{xy} \equiv \frac{\partial \sigma^a}{\partial z^x}\frac{\partial \sigma^b}{\partial z^y}\gamma_{ab} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.15}$$

The integration measure gets a factor of 2 from the Jacobian, $\mathrm{d}^2 z = 2\mathrm{d}^2\sigma$, and since $\sqrt{|\gamma|} = 1/2$ we have

$$\sqrt{|\gamma|}\,\mathrm{d}^2 z = \sqrt{|\gamma|}\,\mathrm{d}^2\sigma. \tag{2.16}$$

In complex coordinates, the action thus becomes

$$S_{\mathrm{P,C}}[X] = \frac{1}{4\pi}\int \mathrm{d}^2 z\, h_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu, \tag{2.17}$$

where the subscript C stands for complex. We shall hereafter write the Polyakov action simply as S; the signature and coordinate system are understood from context.

The equation of motion obtained by varying S with respect to X is $\partial\bar{\partial}X = 0$, from which we conclude that $\partial X(z)$ is holomorphic (left-moving) and $\bar{\partial}X(\bar{z})$ is antiholomorphic (right-moving). As we shall see, this notion of worldsheet chirality will be of great importance in metastring theory.

2.3 The Energy-Momentum Tensor

The energy-momentum tensor of the worldsheet CFT may be calculated by varying the action with respect to the metric:

$$T_{ab} \equiv -\frac{4\pi}{\sqrt{\gamma}} \frac{\delta S}{\delta \gamma^{ab}}, \tag{2.18}$$

where the factor of -4π is just a convention. An alternative definition in terms of frame fields is

$$T_{ab} \equiv \frac{2\pi \gamma_{ac} {e_l}^c}{\det(e)} \frac{\delta S}{\delta {e_l}^b}. \tag{2.19}$$

The calculation is straightforward, and may be found in Appendix A. Both definitions produce the result

$$T_{ab} = -\frac{1}{2} h_{\mu\nu} \left(\partial_a X^\mu \partial_b X^\nu - \frac{1}{2} \gamma_{ab} \partial_c X^\mu \partial^c X^\nu \right). \tag{2.20}$$

In components, we have

$$T_{00} = T_{11} = -\frac{1}{4} h_{\mu\nu} \left(\partial_0 X^\mu \partial_0 X^\nu + \partial_1 X^\mu \partial_1 X^\nu \right), \tag{2.21}$$

$$T_{01} = T_{10} = -\frac{1}{2} h_{\mu\nu} \partial_0 X^\mu \partial_1 X^\nu. \tag{2.22}$$

We should also write the components of the energy-momentum tensor in complex coordinates, using the metric

$$\gamma_{xy} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.23}$$

This gives

$$T(z) \equiv T_{zz} = -\frac{1}{2} h_{\mu\nu} \partial X^\mu \partial X^\nu, \qquad \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}} = -\frac{1}{2} h_{\mu\nu} \bar{\partial} X^\mu \bar{\partial} X^\nu, \tag{2.24}$$

and $T_{z\bar{z}} = T_{z\bar{z}} = 0$, which means the energy-momentum tensor is traceless, as it indeed must be. Since it is also conserved, we have

$$\gamma^{xy} \partial_x T_{yz} = 2\partial_z T_{\bar{z}z} + 2\partial_{\bar{z}} T_{zz} = 2\bar{\partial} T(z) = 0, \tag{2.25}$$

$$\gamma^{xy} \partial_x T_{y\bar{z}} = 2\partial_z T_{\bar{z}\bar{z}} + 2\partial_{\bar{z}} T_{z\bar{z}} = 2\partial \bar{T}(\bar{z}) = 0. \tag{2.26}$$

This means that $T(z)$ is holomorphic and $\bar{T}(\bar{z})$ is antiholomorphic, justifying the notation $T(z)$ and $\bar{T}(\bar{z})$.

2.4 The Propagator

In order to calculate OPEs in string theory, we first need to derive the XX OPE, also known as the propagator. Recall that the Polyakov action in complex coordinates is

$$S = \frac{1}{4\pi} \int \mathrm{d}^2 z \, h_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu. \tag{2.27}$$

The path integral of a total functional derivative vanishes, and thus we may write

$$\begin{aligned} 0 &= \int \mathcal{D}X \frac{\delta}{\delta X_\mu(z,\bar{z})} \left(\mathrm{e}^{-S} X^\nu(w,\bar{w}) \right) \\ &= \int \mathcal{D}X \, \mathrm{e}^{-S} \left(h^{\mu\nu} \delta(z-w, \bar{z}-\bar{w}) + \frac{1}{2\pi} \partial\bar{\partial} X^\mu(z,\bar{z}) X^\nu(w,\bar{w}) \right) \\ &= h^{\mu\nu} \langle \delta(z-w, \bar{z}-\bar{w}) \rangle + \frac{1}{2\pi} \partial\bar{\partial} \langle X^\mu(z,\bar{z}) X^\nu(w,\bar{w}) \rangle, \end{aligned}$$

where $\partial \equiv \partial_z$ and $\bar{\partial} \equiv \partial_{\bar{z}}$. Thus the following holds as an operator equation[4]:

$$\partial\bar{\partial}X^\mu(z,\bar{z})\,X^\nu(w,\bar{w}) = -2\pi h^{\mu\nu}\delta(z-w,\bar{z}-\bar{w})\,. \tag{2.28}$$

It's easy to check that $-h^{\mu\nu}\ln|z-w|^2$ solves this equation. Indeed, for $z \neq w$ we have

$$\partial\bar{\partial}\left(-h^{\mu\nu}\ln|z-w|^2\right) = -h^{\mu\nu}\partial\bar{\partial}\left(\ln(z-w)+\ln(\bar{z}-\bar{w})\right) = 0. \tag{2.29}$$

For $z \to w$ we use the divergence theorem:

$$\int_A\left(\partial\omega^z - \bar{\partial}\omega^{\bar{z}}\right)\mathrm{d}^2z = \frac{\mathrm{i}}{2}\oint_{\partial A}\left(\omega^z\,\mathrm{d}\bar{z} + \omega^{\bar{z}}\,\mathrm{d}z\right), \tag{2.30}$$

where ∂A is a counterclockwise circle around the integration region A. Integrating the left-hand side of the above equation, we obtain

$$\begin{aligned}\int_A\partial\bar{\partial}\left(-h^{\mu\nu}\ln|z-w|^2\right)\mathrm{d}^2z &= -h^{\mu\nu}\int_A\left(\partial\frac{1}{\bar{z}-\bar{w}} + \bar{\partial}\frac{1}{z-w}\right)\mathrm{d}^2z\\ &= \frac{\mathrm{i}}{2}h^{\mu\nu}\oint_{\partial A}\left(\frac{1}{z-w}\mathrm{d}z - \frac{1}{\bar{z}-\bar{w}}\mathrm{d}\bar{z}\right)\\ &= -2\pi h^{\mu\nu}\\ &= -2\pi h^{\mu\nu}\int_A\delta(z-w,\bar{z}-\bar{w})\,\mathrm{d}^2z.\end{aligned}$$

We thus conclude that

$$X^\mu(z,\bar{z})\,X^\nu(w,\bar{w}) = -h^{\mu\nu}\ln|z-w|^2\,. \tag{2.31}$$

2.5 Calculating OPEs

In the quantum theory, the components of the energy-momentum tensor are

$$T(z) = -\frac{1}{2}h_{\mu\nu}:\partial X^\mu\partial X^\nu:,\qquad \bar{T}(\bar{z}) = -\frac{1}{2}h_{\mu\nu}:\bar{\partial}X^\mu\bar{\partial}X^\nu:, \tag{2.32}$$

where the :: indicate *normal ordering*. The OPE of two normal-ordered operators is given by

$$:\mathcal{A}::\mathcal{B}:\equiv:\mathcal{AB}:+\sum\text{contractions}, \tag{2.33}$$

where the sum is over all possible choices of different pairs of fields X^μ, one from $\mathcal{A}$ and one from $\mathcal{B}$, replacing each pair with the contraction given by the propagator found in section 2.4:

$$\langle X^\mu(z,\bar{z})\,X^\nu(w,\bar{w})\rangle = -h^{\mu\nu}\ln|z-w|^2\,. \tag{2.34}$$

We use $\langle\rangle$ to denote the contraction of any two fields. By taking derivatives of both sides, we may derive additional contractions:

$$\langle\partial X^\mu(z)\,\partial X^\nu(w)\rangle = -h^{\mu\nu}\frac{1}{(z-w)^2}, \tag{2.35}$$

$$\langle\bar{\partial}X^\mu(z)\,\bar{\partial}X^\nu(w)\rangle = -h^{\mu\nu}\frac{1}{(\bar{z}-\bar{w})^2}, \tag{2.36}$$

$$\langle\partial X^\mu(z)\,\bar{\partial}X^\nu(\bar{w})\rangle = 0. \tag{2.37}$$

Another important contraction is that of a coordinate field and an exponential, given by the OPE of ∂X^μ with $\mathrm{e}^{\mathrm{i}\,p\cdot X}$. Expanding the exponential in a series, we get:

$$\partial X^\mu(z)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar{w})} = \partial X^\mu(z)\sum_{n=0}^{\infty}\frac{(\mathrm{i}\,p_\nu X^\nu(w,\bar{w}))^n}{n!}. \tag{2.38}$$

[4]An operator equation of the form $F(z,\bar{z}) = 0$ should be interpreted as a statement about expectation values $\langle F(z,\bar{z})\ldots\rangle = 0$, where $\ldots$ stands for arbitrary operator insertions at points $\neq z,\bar{z}$.

For each n, there are exactly n ways to perform a contraction of $\partial X^\mu(z)$ with an $X^\nu(w,\bar{w})$. Such a contraction is of the form

$$\langle \partial X^\mu(z)\, X^\nu(w,\bar{w})\rangle = -h^{\mu\nu}\frac{\partial}{\partial z}\ln|z-w|^2 = -h^{\mu\nu}\frac{1}{z-w}. \tag{2.39}$$

We thus get

$$\begin{aligned} :\partial X^\mu(z)::\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar{w})}: &= \sum_{n=0}^{\infty} n\cdot : \frac{\left(\mathrm{i}\,p_\alpha X^\alpha(w,\bar{w})\right)^{n-1}}{n!} : \mathrm{i}\,p_\nu \langle \partial X^\mu(z)\, X^\nu(w,\bar{w})\rangle \\ &= -\,\mathrm{i}\,h^{\mu\nu}p_\nu \frac{1}{z-w}\sum_{n=0}^{\infty} : \frac{\left(\mathrm{i}\,p_\alpha X^\alpha(w,\bar{w})\right)^{n}}{n!} : \\ &= -\,\mathrm{i}\,h^{\mu\nu}p_\nu \frac{1}{z-w} : \mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar{w})} :, \end{aligned}$$

or

$$\langle \partial X^\mu(z)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar{w})}\rangle = -\,\mathrm{i}\,h^{\mu\nu}p_\nu\frac{1}{z-w}\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar{w})}, \tag{2.40}$$

where it should be understood that the exponential on the right-hand side is not part of the contraction; it remains, untouched, in the original expression, just as it would if one attempted to differentiate or integrate it.

2.6 Perturbations of the Spacetime Metric

The Polyakov action with flat Lorentzian metric is of the form

$$S \sim \int \mathrm{d}^2\sigma\, h_{\mu\nu}\left(\partial_\tau X^\mu \partial_\tau X^\nu - \partial_\sigma X^\mu \partial_\sigma X^\nu\right). \tag{2.41}$$

From this action we may find the momentum conjugate to X:

$$P_\mu \equiv \frac{\delta S}{\delta\left(\partial_\tau X^\mu\right)} = h_{\mu\nu}\partial_\tau X^\nu. \tag{2.42}$$

Recall that the components of the energy-momentum tensor are

$$T(z) = -\frac{1}{2}h_{\mu\nu}\partial X^\mu \partial X^\nu, \qquad \bar{T}(\bar{z}) = -\frac{1}{2}h_{\mu\nu}\bar{\partial} X^\mu \bar{\partial} X^\nu. \tag{2.43}$$

The holomorphic and antiholomorphic derivatives are given by

$$\partial X^\mu = \frac{1}{2}\left(\partial_\sigma X^\mu + \partial_\tau X^\mu\right) = \frac{1}{2}\left(\partial_\sigma X^\mu + h^{\mu\nu}P_\nu\right), \tag{2.44}$$

$$\bar{\partial} X^\mu = \frac{1}{2}\left(\partial_\sigma X^\mu - \partial_\tau X^\mu\right) = \frac{1}{2}\left(\partial_\sigma X^\mu - h^{\mu\nu}P_\nu\right). \tag{2.45}$$

Note the appearance of the (inverse) metric, $h^{\mu\nu}$, in these expressions. We can invert them to find

$$\partial_\sigma X^\mu = \partial X^\mu + \bar{\partial} X^\mu, \qquad P_\nu = h_{\mu\nu}\left(\partial X^\mu - \bar{\partial} X^\mu\right). \tag{2.46}$$

Now, we perturb $h_{\mu\nu} \mapsto h_{\mu\nu} + \delta h_{\mu\nu}$. Then

$$\delta\left(\partial X^\mu\right) = +\frac{1}{2}\delta h^{\mu\nu}P_\nu = +\frac{1}{2}\delta h^{\mu\nu}h_{\alpha\nu}\left(\partial X^\alpha - \bar{\partial} X^\alpha\right), \tag{2.47}$$

$$\delta\left(\bar{\partial} X^\mu\right) = -\frac{1}{2}\delta h^{\mu\nu}P_\nu = -\frac{1}{2}\delta h^{\mu\nu}h_{\alpha\nu}\left(\partial X^\alpha - \bar{\partial} X^\alpha\right). \tag{2.48}$$

Using the identity[5]

$$h_{\alpha\beta}\delta h^{\beta\gamma} = -\delta h_{\alpha\beta}h^{\beta\gamma}, \tag{2.49}$$

[5] This comes from $h_{\alpha\beta}\delta h^{\beta\gamma} + \delta h_{\alpha\beta}h^{\beta\gamma} = \delta\left(h_{\alpha\beta}h^{\beta\gamma}\right) = \delta\left(\delta_\alpha^\gamma\right) = 0$.

we discover that the energy-momentum tensor changes as follows:

$$\begin{aligned}\delta T &= -\frac{1}{2}\delta h_{\mu\nu}\partial X^\mu \partial X^\nu - h_{\mu\nu}\partial X^\mu \delta\left(\partial X^\nu\right)\\ &= -\frac{1}{2}\delta h_{\mu\nu}\partial X^\mu \partial X^\nu - \frac{1}{2}h_{\mu\nu}\partial X^\mu \delta h^{\nu\alpha}h_{\alpha\beta}\left(\partial X^\beta - \bar{\partial}X^\beta\right)\\ &= -\frac{1}{2}\delta h_{\mu\nu}\partial X^\mu \partial X^\nu + \frac{1}{2}\delta h_{\mu\nu}\partial X^\mu\left(\partial X^\nu - \bar{\partial}X^\nu\right)\\ &= -\frac{1}{2}\delta h_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu,\end{aligned}$$

$$\begin{aligned}\delta \bar{T} &= -\frac{1}{2}\delta h_{\mu\nu}\bar{\partial} X^\mu \bar{\partial} X^\nu - h_{\mu\nu}\bar{\partial} X^\mu \delta\left(\bar{\partial} X^\nu\right)\\ &= -\frac{1}{2}\delta h_{\mu\nu}\bar{\partial} X^\mu \bar{\partial} X^\nu + \frac{1}{2}h_{\mu\nu}\bar{\partial} X^\mu \delta h^{\nu\alpha}h_{\alpha\beta}\left(\partial X^\beta - \bar{\partial}X^\beta\right)\\ &= -\frac{1}{2}\delta h_{\mu\nu}\bar{\partial} X^\mu \bar{\partial} X^\nu - \frac{1}{2}\delta h_{\mu\nu}\bar{\partial} X^\mu\left(\partial X^\nu - \bar{\partial}X^\nu\right)\\ &= -\frac{1}{2}\delta h_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu.\end{aligned}$$

In particular, we see that $\delta T = \delta \bar{T}$. According to our definitions in section 2.1, we see that these perturbations define an (achiral) canonical deformation if they are primary fields of weight $(1,1)$.

2.7 Deforming the Worldsheet CFT

Let us denote $V_{\mu\nu} \equiv \delta h_{\mu\nu}$. Based on the analysis of section 2.6, we define

$$\delta T \equiv \delta \bar{T} \equiv V_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu, \tag{2.50}$$

and demand that the deformation is canonical, that is, δT and $\delta \bar{T}$ are $(1,1)$ primary fields. This will ensure that conformal symmetry is not broken. Of course, since $\delta T = \delta \bar{T}$, it's enough to consider only δT. In order to determine the necessary conditions on $V_{\mu\nu}$ for δT to be of weight $(1,1)$, we must calculate its OPE with T.

To facilitate this calculation, we expand $V_{\mu\nu}$ as a superposition of plane waves, or in other words, we perform a Fourier transform:

$$V_{\mu\nu}\left(X\right) \equiv \int \frac{\mathrm{d}^D p}{\left(2\pi\right)^D}\Pi_{\mu\nu}\left(p\right)\mathrm{e}^{\mathrm{i}\,p\cdot X}, \tag{2.51}$$

where $p \cdot X \equiv p_\mu X^\mu$ is the spacetime (Minkowski) inner product, p_μ is the string's D-momentum and $\Pi_{\mu\nu}$ is a polarization tensor. Note that p_μ and $\Pi_{\mu\nu}\left(p\right)$ are constant on the worldsheet, that is, they don't depend on $z, \bar{z}$. We assume that $\Pi_{\mu\nu}$ is symmetric since we are only interested in perturbations of the spacetime metric. Now we can focus on a single plane wave of momentum p:[6]

$$\delta T = \Pi_{\mu\nu}\,\mathrm{e}^{\mathrm{i}\,p\cdot X}\,\partial X^\mu \bar{\partial} X^\nu. \tag{2.52}$$

Let us calculate the OPE of T and δT:

$$T\left(z\right)\delta T\left(w,\bar{w}\right) = -\frac{1}{2}h_{\mu\nu}\Pi_{\alpha\beta} : \partial X^\mu\left(z\right)\partial X^\nu\left(z\right) :: \partial X^\alpha\left(w\right)\bar{\partial} X^\beta\left(\bar{w}\right)\mathrm{e}^{\mathrm{i}\,p\cdot X\left(w,\bar{w}\right)} : . \tag{2.53}$$

[6]The astute reader will recognize a similarity with the graviton vertex operator, $\int \mathrm{d}^2 z\,\Pi_{\mu\nu}\,\mathrm{e}^{\mathrm{i}\,p\cdot X}\,\partial X^\mu \bar{\partial} X^\nu$, corresponding to the first excited state of a closed string. This is, of course, not a coincidence, as we have briefly mentioned in section 2.1. However, we will not discuss the string spectrum in this work.

First, we sum on all possible contractions:

$$
\begin{aligned}
&T(z)\,\delta T(w,\bar w) = \\
&-\frac{1}{2}h_{\mu\nu}\Pi_{\alpha\beta} : \bar\partial X^{\beta}(\bar w)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)} : \langle \partial X^{\mu}(z)\,\partial X^{\alpha}(w)\rangle\langle \partial X^{\nu}(z)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)}\rangle + (\mu\leftrightarrow\nu) + \\
&-\frac{1}{2}h_{\mu\nu}\Pi_{\alpha\beta} : \partial X^{\alpha}(w)\,\bar\partial X^{\beta}(\bar w)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)} : \langle \partial X^{\mu}(z)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)}\rangle\langle \partial X^{\nu}(z)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)}\rangle + \\
&-\frac{1}{2}h_{\mu\nu}\Pi_{\alpha\beta} : \partial X^{\nu}(z)\,\bar\partial X^{\beta}(\bar w)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)} : \langle \partial X^{\mu}(z)\,\partial X^{\alpha}(w)\rangle + (\mu\leftrightarrow\nu) + \\
&-\frac{1}{2}h_{\mu\nu}\Pi_{\alpha\beta} : \partial X^{\nu}(z)\,\partial X^{\alpha}(w)\,\bar\partial X^{\beta}(\bar w)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)} : \langle \partial X^{\mu}(z)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)}\rangle + (\mu\leftrightarrow\nu)\,.
\end{aligned}
$$

Writing the contractions explicitly, we get

$$
\begin{aligned}
T(z)\,\delta T(w,\bar w) = &-\frac{\mathrm{i}\,h^{\alpha\gamma}p_{\gamma}}{(z-w)^3}\Pi_{\alpha\beta} : \bar\partial X^{\beta}(\bar w)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)} : + \\
&+\frac{p^2/2}{(z-w)^2}\Pi_{\alpha\beta} : \partial X^{\alpha}(w)\,\bar\partial X^{\beta}(\bar w)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)} : + \\
&+\frac{1}{(z-w)^2}\Pi_{\alpha\beta} : \partial X^{\alpha}(z)\,\bar\partial X^{\beta}(\bar w)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)} : + \\
&+\frac{\mathrm{i}\,p_{\mu}}{z-w}\Pi_{\alpha\beta} : \partial X^{\mu}(z)\,\partial X^{\alpha}(w)\,\bar\partial X^{\beta}(\bar w)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)} : .
\end{aligned}
$$

Next, we expand $\partial X^{\mu}(z)$ around $z=w$ and disregard terms nonsingular as $z\to w$:

$$
\begin{aligned}
&T(z)\,\delta T(w,\bar w) = \\
&-\frac{\mathrm{i}\,h^{\alpha\gamma}p_{\gamma}}{(z-w)^3}\Pi_{\alpha\beta} : \bar\partial X^{\beta}(\bar w)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)} : + \frac{1+p^2/2}{(z-w)^2}\Pi_{\alpha\beta} : \partial X^{\alpha}(w)\,\bar\partial X^{\beta}(\bar w)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)} : + \\
&+\frac{1}{z-w}\Pi_{\alpha\beta}\left(\mathrm{i}\,p_{\mu} : \partial X^{\mu}(w)\,\partial X^{\alpha}(w)\,\bar\partial X^{\beta}(\bar w)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)} : + : \partial\partial X^{\alpha}(w)\,\bar\partial X^{\beta}(\bar w)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)} :\right).
\end{aligned}
$$

Recognizing the last line as a derivative

$$
\partial\left(: \partial X^{\alpha}(w)\,\bar\partial X^{\beta}(\bar w)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)} :\right), \tag{2.54}
$$

and plugging in the original expression for δT, we obtain

$$
T(z)\,\delta T(w,\bar w) = -\frac{\mathrm{i}\,h^{\alpha\gamma}p_{\gamma}}{(z-w)^3}\Pi_{\alpha\beta} : \bar\partial X^{\beta}(\bar w)\,\mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar w)} : + \frac{\left(1+p^2/2\right)\delta T(w,\bar w)}{(z-w)^2} + \frac{\partial\delta T(w,\bar w)}{z-w}. \tag{2.55}
$$

The OPE with $\bar T$ may be calculated analogously. From the definition (2.2) of a primary field we see that δT is a primary field with weight $(1,1)$ only of two conditions are satisfied. First, since the weight is

$$
h = \bar h = 1 + \frac{p^2}{2}, \tag{2.56}
$$

we must have $p^2=0$, i.e. the field is massless. Second, in order to get rid of the $(z-w)^{-3}$ term we must have

$$
p_{\mu}\Pi^{\mu\nu} = 0. \tag{2.57}
$$

These two conditions may alternatively be written in terms of a single plane wave $\delta h_{\mu\nu} = V_{\mu\nu} = \Pi_{\mu\nu}\,\mathrm{e}^{\mathrm{i}\,p\cdot X}$ as follows:

$$
\Box\delta h_{\mu\nu} = 0, \qquad \partial_{\mu}\delta h^{\mu\nu} = 0, \tag{2.58}
$$

where $\Box \equiv \partial_{\mu}\partial^{\mu}$ is the spacetime d'Alembertian.

In conclusion, by considering canonical deformations of the worldsheet CFT, we have discovered that the perturbation of the metric, $\delta h_{\mu\nu}$, must obey the linearized Einstein equation, albeit in a particular gauge. As shown in [6], it is possible to obtain a gauge-invariant equation of motion (and much more) by considering more general deformations. This procedure, and its generalization to metastring theory, will be the focus of future investigation.

3 Conformal Deformations in Metastring Theory: Part I

We now turn our attention to metastring theory. We would like to find the equations of motion for the background fields – the so-called *metagravity equations* – by generalizing the procedure used above. However, we will soon learn that such a generalization is not quite straightforward.

First, let us introduce the theory, derive similar results to those derived above for standard string theory, and discuss some features unique to the metastring formulation such as target space chirality and the metageometry.

3.1 The Metastring Worldsheet Theory

3.1.1 The Momentum-Space Polyakov Action

Let us derive the metastring action. The first step is to write down the Polyakov action in terms of momentum-space coordinates. We introduce a momentum scale ε and a length scale λ such that

$$\hbar \equiv \lambda\varepsilon, \qquad \alpha' \equiv \frac{\lambda}{\varepsilon} \quad \Longrightarrow \quad \lambda^2 = \hbar\alpha', \qquad \varepsilon^2 = \frac{\hbar}{\alpha'}. \tag{3.1}$$

Then we may write the Polyakov action (2.5) as a dimensionless first-order action:

$$\hat{S}\left[X, \mathbf{P}\right] \equiv \frac{1}{2\pi}\int_\Sigma \left(\frac{1}{\lambda\varepsilon}\mathbf{P}_\mu \wedge \mathrm{d}X^\mu + \frac{1}{2\varepsilon^2}h^{\mu\nu}\left(\star\mathbf{P}_\mu \wedge \mathbf{P}_\nu\right)\right), \tag{3.2}$$

where $\mathbf{P}$ is an auxiliary 1-form with dimension of momentum. To see that this action is equivalent to (2.5), we should integrate out $\mathbf{P}$. Varying the action with respect to $\mathbf{P}$, we get

$$\delta\hat{S} = \frac{1}{2\pi}\int_\Sigma \left(-\frac{1}{\lambda\varepsilon}\mathrm{d}X^\mu \wedge \delta\mathbf{P}_\mu + \frac{1}{\varepsilon^2}h^{\mu\nu}\left(\star\mathbf{P}_\nu \wedge \delta\mathbf{P}_\mu\right)\right), \tag{3.3}$$

and thus

$$\mathbf{P}_\mu = \frac{\varepsilon}{\lambda}h_{\mu\nu}\left(\star\mathrm{d}X^\nu\right). \tag{3.4}$$

Plugging this back into the action (3.2), it's easy to see that we indeed obtain (2.5) back, or more precisely that

$$\hat{S}\left[X\right] = \frac{1}{4\pi\lambda^2}\int_\Sigma h_{\mu\nu}\left(\star\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu\right) = \frac{\alpha'}{\lambda^2}S_\mathrm{P}\left[X\right] = \frac{1}{\hbar}S_\mathrm{P}\left[X\right], \tag{3.5}$$

where we continue to take $\alpha' \equiv 2$ as before. Alternatively, we can write the equation of motion (3.4) as

$$\mathrm{d}X^\mu = \frac{\lambda}{\varepsilon}h^{\mu\nu}\left(\star\mathbf{P}_\nu\right), \tag{3.6}$$

and plug it into (3.2) to obtain

$$\hat{S}\left[\mathbf{P}\right] = -\frac{1}{4\pi\varepsilon^2}\int_\Sigma h^{\mu\nu}\left(\star\mathbf{P}_\mu \wedge \mathbf{P}_\nu\right). \tag{3.7}$$

Furthermore, by integrating out X, it's easy to see that we get $\mathrm{d}\mathbf{P} = 0$, i.e. $\mathbf{P}$ is closed. Thus we can locally write $\mathbf{P} \equiv \mathrm{d}Y$ where Y is a 0-form with dimensions of momentum; we may think of it as a momentum-space coordinate, as in some sense it locally encodes the degrees of freedom of $\mathbf{P}$. We may then plug $\mathbf{P} = \mathrm{d}Y$ into (3.2) to obtain the Polyakov action for the momentum-space coordinates Y plus a boundary term:

$$\begin{aligned}\hat{S}\left[X, Y\right] &= \frac{1}{2\pi}\int_\Sigma \left(\frac{1}{\lambda\varepsilon}\mathrm{d}Y_\mu \wedge \mathrm{d}X^\mu + \frac{1}{2\varepsilon^2}h^{\mu\nu}\left(\star\mathrm{d}Y_\mu \wedge \mathrm{d}Y_\nu\right)\right) \\ &= \frac{1}{2\pi\lambda\varepsilon}\int_{\partial\Sigma} Y_\mu \mathrm{d}X^\mu + \frac{\alpha'}{\varepsilon^2}S_\mathrm{P}\left[Y\right].\end{aligned}$$

Observe that

$$\hat{S}\left[X, Y\right] = \frac{\alpha'}{\lambda^2}S_\mathrm{P}\left[X\right] \sim \frac{\alpha'}{\varepsilon^2}S_\mathrm{P}\left[Y\right]. \tag{3.8}$$

In this way, the momentum scale ε is replacing the length scale λ in the dual theory.

3.1.2 The Phase Space Metastring Action

The worldsheet of a metastring is a nonlinear sigma model with a $2D$-dimensional phase space as the target manifold. To formulate the appropriate action, we decompose $\mathbf{P}$ in local coordinates as

$$\mathbf{P}_\mu \equiv P_\mu \, \mathrm{d}\sigma + Q_\mu \, \mathrm{d}\tau. \tag{3.9}$$

Then the action (3.2) reads

$$\hat{S}\left[X, Q, P\right] = \frac{1}{2\pi} \int \mathrm{d}^2\sigma \left(\frac{1}{\lambda\varepsilon} \left(P_\mu \partial_\tau X^\mu - Q_\mu \partial_\sigma X^\mu \right) + \frac{1}{2\varepsilon^2} h^{\mu\nu} \left(Q_\mu Q_\nu - P_\mu P_\nu \right) \right), \tag{3.10}$$

where we used $\mathrm{d}X = \partial_\sigma X \, \mathrm{d}\sigma + \partial_\tau X \, \mathrm{d}\tau$ and $\mathrm{d}^2\sigma \equiv \mathrm{d}\sigma \wedge \mathrm{d}\tau$. It's easy to see that the equations of motion for P and Q are

$$P_\mu = \frac{\varepsilon}{\lambda} h_{\mu\nu} \partial_\tau X^\nu, \qquad Q_\mu = \frac{\varepsilon}{\lambda} h_{\mu\nu} \partial_\sigma X^\nu. \tag{3.11}$$

Of course, this also follows directly from our earlier result that $\star\mathbf{P}_\mu = \frac{\varepsilon}{\lambda} h_{\mu\nu} \, \mathrm{d}X^\nu$. Let us integrate out Q by inserting its equation of motion into the action:

$$\hat{S}\left[X, P\right] = \frac{1}{2\pi} \int \mathrm{d}^2\sigma \left(\frac{1}{\lambda\varepsilon} P_\mu \partial_\tau X^\mu - \frac{1}{2\lambda^2} h_{\mu\nu} \partial_\sigma X^\mu \partial_\sigma X^\nu - \frac{1}{2\varepsilon^2} h^{\mu\nu} P_\mu P_\nu \right). \tag{3.12}$$

Recalling our earlier definition $\mathbf{P} \equiv \mathrm{d}Y$, we take $P \equiv \partial_\sigma Y$ where Y is a momentum-space coordinate. Then the action becomes

$$\hat{S}\left[X, Y\right] = \frac{1}{2\pi} \int \mathrm{d}^2\sigma \left(\frac{1}{\lambda\varepsilon} \partial_\tau X^\mu \partial_\sigma Y_\mu - \frac{1}{2\lambda^2} h_{\mu\nu} \partial_\sigma X^\mu \partial_\sigma X^\nu - \frac{1}{2\varepsilon^2} h^{\mu\nu} \partial_\sigma Y_\mu \partial_\sigma Y_\nu \right). \tag{3.13}$$

Now comes the moment we have been building up for. We unify X and Y into a dimensionless coordinate on phase space which, in a notation borrowed from double field theory[7], is defined as

$$\mathbb{X}^A \equiv \begin{pmatrix} X^\mu/\lambda \\ Y_\mu/\varepsilon \end{pmatrix}. \tag{3.14}$$

From now on we will always take $\lambda \equiv \varepsilon \equiv 1$ for brevity, such that $\mathbb{X} \equiv (X, Y)$. The phase space $\mathcal{P}$, with coordinates $\mathbb{X}^A$, has $2D$ dimensions. In terms of indices, if spacetime has D dimensions then $\mu = 1, \ldots, D$ while $A = 1, \ldots, 2D$. In order to write the action in terms of $\mathbb{X}$ we must have a way to project it onto X and Y. For this purpose we define a constant *polarization metric* (or *P-metric*) η of signature (D, D):

$$\eta_{AB} \equiv \begin{pmatrix} 0 & \delta \\ \delta^{\mathrm{T}} & 0 \end{pmatrix}, \tag{3.15}$$

where δ is the D-dimensional identity matrix, such that

$$\mathbb{X}^A \eta_{AB} \mathbb{X}'^B = \begin{pmatrix} X^\mu & Y_\mu \end{pmatrix} \begin{pmatrix} 0 & \delta_\mu^\nu \\ \delta_\nu^\mu & 0 \end{pmatrix} \begin{pmatrix} X'^\nu \\ Y'_\nu \end{pmatrix} = X^\mu Y'_\mu + Y_\mu X'^\mu. \tag{3.16}$$

We also define a constant *quantum metric* (or *Q-metric*) H of signature $(2, 2D-2)$:

$$H_{AB} \equiv \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}, \tag{3.17}$$

where $h_{\mu\nu}$ is the D-dimensional Lorentzian metric, such that

$$\mathbb{X}^A H_{AB} \mathbb{X}'^B = \begin{pmatrix} X^\mu & Y_\mu \end{pmatrix} \begin{pmatrix} h_{\mu\nu} & 0 \\ 0 & h^{\mu\nu} \end{pmatrix} \begin{pmatrix} X'^\nu \\ Y'_\nu \end{pmatrix} = X^\mu h_{\mu\nu} X'^\nu + Y_\mu h^{\mu\nu} Y'_\nu. \tag{3.18}$$

Finally, we define a constant symplectic form ω:

$$\omega_{AB} \equiv \begin{pmatrix} 0 & \delta \\ -\delta^{\mathrm{T}} & 0 \end{pmatrix}, \tag{3.19}$$

[7]For a review see, for example, [10].

such that

$$\mathbb{X}^A \omega_{AB} \mathbb{X}'^B = \begin{pmatrix} X^\mu & Y_\mu \end{pmatrix} \begin{pmatrix} 0 & \delta^\nu_\mu \\ -\delta^\mu_\nu & 0 \end{pmatrix} \begin{pmatrix} X'^\nu \\ Y_{\prime\nu} \end{pmatrix} = X^\mu Y'_\mu - Y_\mu X'^\mu. \tag{3.20}$$

It's easy to see that the last two terms in the action (3.13) may be written succinctly using the Q-metric H and the phase space coordinates $\mathbb{X}$ as

$$\frac{1}{2\varepsilon^2} h^{\mu\nu} \partial_\sigma Y_\mu \partial_\sigma Y_\nu + \frac{1}{2\lambda^2} h_{\mu\nu} \partial_\sigma X^\mu \partial_\sigma X^\nu = \frac{1}{2} \partial_\sigma \mathbb{X}^A H_{AB} \partial_\sigma \mathbb{X}^B. \tag{3.21}$$

To write the first term in (3.13) in terms of $\mathbb{X}$ we note that the combination $\frac{1}{2}(\eta + \omega)$ projects $\mathbb{X} \equiv (X, Y)$ onto $(Y, 0)$ while $\frac{1}{2}(\eta - \omega)$ projects it onto $(0, X)$:

$$\frac{1}{2} (\eta + \omega)_{AB} \mathbb{X}^B = \begin{pmatrix} 0 & \delta^\nu_\mu \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X^\nu \\ Y_\nu \end{pmatrix} = \begin{pmatrix} Y_\mu \\ 0 \end{pmatrix}, \tag{3.22}$$

$$\frac{1}{2} (\eta - \omega)_{AB} \mathbb{X}^B = \begin{pmatrix} 0 & 0 \\ \delta^\mu_\nu & 0 \end{pmatrix} \begin{pmatrix} X^\nu \\ Y_\nu \end{pmatrix} = \begin{pmatrix} 0 \\ X^\mu \end{pmatrix}. \tag{3.23}$$

We can thus write

$$\frac{1}{\lambda\varepsilon} \partial_\tau X^\mu \partial_\sigma Y_\mu = \frac{1}{2} \partial_\tau \mathbb{X}^A (\eta + \omega)_{AB} \partial_\sigma \mathbb{X}^B. \tag{3.24}$$

This allows us to finally write the Lorentzian metastring sigma-model action, also known as the *Tseytlin action*:

$$S[\mathbb{X}] \equiv \frac{1}{4\pi} \int \mathrm{d}^2\sigma \left(\partial_\tau \mathbb{X}^A (\eta + \omega)_{AB} \partial_\sigma \mathbb{X}^B - \partial_\sigma \mathbb{X}^A H_{AB} \partial_\sigma \mathbb{X}^B \right). \tag{3.25}$$

Together, the metrics H and η and the symplectic form ω describe the geometry of the metastring target space, which we shall refer to as the *metageometry*. In this work we will take them to be a flat and fixed background, and find the behavior of small perturbations over this fixed background. However, in general the metageometry can and should be completely dynamical. The theory describing the dynamics of the metageometry is called *metagravity*.

The symplectic structure ω expresses the fact that our phase space $\mathcal{P}$ is a symplectic manifold. Both spacetime L and momentum space $\tilde{L}$ are D-dimensional Lagrangian submanifolds of the $2D$-dimensional phase space. Indeed, it's easy to see that ω vanishes on pure spacetime vectors of the form $(X, 0)$ as well as pure momentum-space vectors of the form $(0, Y)$.

Furthermore, as may be seen from equations (3.22) and (3.23) above, the kernel of $\eta + \omega$ $(\eta - \omega)$ consists of pure spacetime (momentum-space) vectors: $L \equiv \ker(\eta + \omega)$ and $\tilde{L} \equiv \ker(\eta - \omega)$. Together, the spacetime and momentum-space manifolds make up a bilagrangian structure on $\mathcal{P}$, decomposing it into two transverse Lagrangian submanifolds such that $T\mathcal{P} = TL \oplus T\tilde{L}$ and $TL \cap T\tilde{L} = \{0\}$.

The P-metric η has the property that $\mathbb{X}^A \eta_{AB} \mathbb{X}^B = 0$ on L and $\tilde{L}$, that is, they are null subspaces of η. If we allow η to be arbitrary, it changes the definition of spacetime and momentum space. The choice of L and $\tilde{L}$ as Lagrangian submanifolds of $\mathcal{P}$ is called a *polarization*, and thus η is called the polarization metric.

The Q-metric H is the generalized metric on $\mathcal{P}$; when allowed to be dynamical, it endows phase space with a dynamical metric structure[8]. When restricted to the spacetime manifold L it reduces to the usual spacetime metric, $h \equiv H\big|_L$.

[8]Note that in a more general background containing a 2-form B-field, H will depend on B. However, as previously stated, in this work we are assuming a vanishing B-field.

3.1.3 The Metastring Energy-Momentum Tensor

Our next step is to find the energy-momentum tensor for the metastring worldsheet. Since our action is not symmetric in τ, σ, we use the frame field method introduced in 2.3 and Appendix A. We define the frame fields ∂_l and co-frame fields e^l by

$$e^l \equiv e^l{}_a \, \mathrm{d}\sigma^a, \qquad \partial_l \equiv e_l{}^a \partial_a, \qquad g_{lm} e^l{}_a e^m{}_b = \gamma_{ab}, \qquad \gamma_{ab} e_l{}^a e_m{}^b = g_{lm}, \qquad \partial_l e^m = \delta_l^m. \tag{3.26}$$

We take $e^l{}_a = \delta^l_a$ and $e_l{}^a = \delta^a_l$. Thus we may write the Lorentzian metastring action as

$$S = \frac{1}{4\pi} \int \mathrm{d}^2\sigma \, \det(e) \left(\partial_0 \mathbb{X}^A (\eta + \omega)_{AB} \, \partial_1 \mathbb{X}^B - \partial_1 \mathbb{X}^A H_{AB} \partial_1 \mathbb{X}^B \right), \tag{3.27}$$

where $0, 1$ are internal indices. Recall that the energy-momentum tensor is given by

$$T_{ab} = \gamma_{ac} e_l{}^c T^l{}_b = \frac{2\pi \gamma_{ac} e_l{}^c}{\det(e)} \frac{\delta S}{\delta e_l{}^b}. \tag{3.28}$$

To calculate this, we use

$$\delta \det(e) = -\det(e) \, e^l{}_a \delta e_l{}^a, \qquad \delta \partial_l = \delta e_l{}^a e^m{}_a \partial_m, \tag{3.29}$$

to get

$$\begin{aligned} \delta S = &-\frac{1}{4\pi} \int \mathrm{d}^2\sigma \, \det(e) \left(\partial_0 \mathbb{X}^A (\eta + \omega)_{AB} \, \partial_1 \mathbb{X}^B - \partial_1 \mathbb{X}^A H_{AB} \partial_1 \mathbb{X}^B \right) e^l{}_b \delta e_l{}^b + \\ &+ \frac{1}{4\pi} \int \mathrm{d}^2\sigma \, \det(e) \, \partial_l \mathbb{X}^A (\eta + \omega)_{AB} \, \partial_1 \mathbb{X}^B e^l{}_b \delta e_0{}^b + \\ &+ \frac{1}{4\pi} \int \mathrm{d}^2\sigma \, \det(e) \left(\partial_0 \mathbb{X}^A (\eta + \omega)_{AB} \, \partial_l \mathbb{X}^B - 2 \partial_1 \mathbb{X}^A H_{AB} \partial_l \mathbb{X}^B \right) e^l{}_b \delta e_1{}^b. \end{aligned}$$

Therefore

$$\begin{aligned} T_{ab} &= \frac{2\pi \gamma_{ac} e_l{}^c}{\det(e)} \frac{\delta S}{\delta e_l{}^b} \\ &= \frac{1}{2} (\eta + \omega)_{AB} \left(\partial_b \mathbb{X}^A \partial_1 \mathbb{X}^B \gamma_{a0} + \partial_0 \mathbb{X}^A \partial_b \mathbb{X}^B \gamma_{a1} - \partial_0 \mathbb{X}^A \partial_1 \mathbb{X}^B \gamma_{ab} \right) + \\ &\quad + \frac{1}{2} H_{AB} \left(\partial_1 \mathbb{X}^A \partial_1 \mathbb{X}^B \gamma_{ab} - 2 \partial_1 \mathbb{X}^A \partial_b \mathbb{X}^B \gamma_{a1} \right). \end{aligned}$$

The individual components are:

$$T_{00} = -\frac{1}{2} \partial_1 \mathbb{X}^A H_{AB} \partial_1 \mathbb{X}^B, \qquad T_{01} = -\frac{1}{2} \partial_1 \mathbb{X}^A (\eta + \omega)_{AB} \, \partial_1 \mathbb{X}^B, \tag{3.30}$$

$$T_{10} = -\partial_0 \mathbb{X}^A H_{AB} \partial_1 \mathbb{X}^B + \frac{1}{2} \partial_0 \mathbb{X}^A (\eta + \omega)_{AB} \, \partial_0 \mathbb{X}^B, \qquad T_{11} = -\frac{1}{2} \partial_1 \mathbb{X}^A H_{AB} \partial_1 \mathbb{X}^B. \tag{3.31}$$

For brevity we denote contraction using the P-metric η by $\cdot$ and define the chiral structure $J \equiv \eta^{-1} H$, such that for two phase-space vectors V, W we have

$$V \cdot W \equiv V^A \eta_{AB} W^B, \qquad V \cdot JW \equiv V^A H_{AB} W^B. \tag{3.32}$$

Noting in addition that ω is antisymmetric, and thus vanishes when contracted with two identical vectors, we may write:

$$T_{00} = -\frac{1}{2} \partial_1 \mathbb{X} \cdot J \partial_1 \mathbb{X}, \qquad T_{01} = -\frac{1}{2} \partial_1 \mathbb{X} \cdot \partial_1 \mathbb{X}, \tag{3.33}$$

$$T_{10} = -\partial_0 \mathbb{X} \cdot J \partial_1 \mathbb{X} + \frac{1}{2} \partial_0 \mathbb{X} \cdot \partial_0 \mathbb{X}, \qquad T_{11} = -\frac{1}{2} \partial_1 \mathbb{X} \cdot J \partial_1 \mathbb{X}. \tag{3.34}$$

Observe that the energy-momentum tensor is traceless, $T^a_a = 0$, as expected.

3.1.4 The Lorentz Constraint and Gauge Fixing

Recall that the worldsheet metric is the Minkowski metric $\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \mathrm{d}\sigma^2$. An infinitesimal Lorentz transformation on the worldsheet with parameter λ is of the form $\delta\,\mathrm{d}\tau = \lambda\,\mathrm{d}\sigma$ and $\delta\,\mathrm{d}\sigma = \lambda\,\mathrm{d}\tau$. This gives

$$\delta\left(\partial_\tau \mathbb{X}\right) = \frac{1}{\lambda}\partial_\sigma \mathbb{X}, \qquad \delta\left(\partial_\sigma \mathbb{X}\right) = \frac{1}{\lambda}\partial_\tau \mathbb{X}. \tag{3.35}$$

Let us write down the variation of the action (3.25) with respect to this transformation:

$$\delta S = \frac{1}{2\pi}\int \mathrm{d}^2\sigma \frac{1}{\lambda}\left(\frac{1}{2}\left(\partial_\sigma \mathbb{X}^A \eta_{AB}\partial_\sigma \mathbb{X}^B + \partial_\tau \mathbb{X}^A \eta_{AB}\partial_\tau \mathbb{X}^B\right) - \partial_\tau \mathbb{X}^A H_{AB}\partial_\sigma \mathbb{X}^B\right), \tag{3.36}$$

where we used the fact that ω is antisymmetric. Thus the action is Lorentz-invariant if the following (Lorentz) constraint is satisfied:

$$\hat{L} \equiv \frac{1}{2}\left(\partial_\sigma \mathbb{X}^A \eta_{AB}\partial_\sigma \mathbb{X}^B + \partial_\tau \mathbb{X}^A \eta_{AB}\partial_\tau \mathbb{X}^B\right) - \partial_\tau \mathbb{X}^A H_{AB}\partial_\sigma \mathbb{X}^B = 0. \tag{3.37}$$

With the η inner product $\cdot$ defined above and $J \equiv \eta^{-1}H$, we may write $\hat{L}$ as

$$\hat{L} = \frac{1}{2}\left(\partial_\tau \mathbb{X}\cdot\partial_\tau \mathbb{X} + \partial_\sigma \mathbb{X}\cdot\partial_\sigma \mathbb{X}\right) - \partial_\tau \mathbb{X}\cdot J\partial_\sigma \mathbb{X}. \tag{3.38}$$

Let us now define the phase-space vector

$$\mathbb{S} \equiv \partial_\tau \mathbb{X} - J\partial_\sigma \mathbb{X}. \tag{3.39}$$

Then

$$\frac{1}{2}\mathbb{S}\cdot\mathbb{S} = \frac{1}{2}\left(\partial_\tau \mathbb{X}\cdot\partial_\tau \mathbb{X} + \partial_\sigma \mathbb{X}\cdot J^2\partial_\sigma \mathbb{X}\right) - \partial_\tau \mathbb{X}\cdot J\partial_\sigma \mathbb{X}, \tag{3.40}$$

and thus we may write

$$\hat{L} = \frac{1}{2}\mathbb{S}\cdot\mathbb{S} + \frac{1}{2}\partial_\sigma \mathbb{X}\cdot\left(1 - J^2\right)\partial_\sigma \mathbb{X}. \tag{3.41}$$

We see that this constraint imposes $J^2 = 1$ and $\mathbb{S}\cdot\mathbb{S} = 0$. Now, the equation of motion may be found by varying (3.25) with respect to $\mathbb{X}$. We have

$$\delta S = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma\left(\partial_\tau\partial_\sigma \mathbb{X}^A \eta_{AB} - \partial_\sigma\partial_\sigma \mathbb{X}^A H_{AB}\right)\delta\mathbb{X}^B, \tag{3.42}$$

which gives

$$\partial_\sigma\left(\left(\eta\partial_\tau - H\partial_\sigma\right)\mathbb{X}\right) = 0 \quad\Longrightarrow\quad \partial_\sigma \mathbb{S} = 0. \tag{3.43}$$

Hence $\mathbb{S}$ depends only on τ. The Lorentz condition $\mathbb{S}\cdot\mathbb{S} = 0$ means that $\mathbb{S}\left(\tau\right)$ is in a null subspace of the P-metric η and thus in a Lagrangian subspace $\tilde{L}$ of $\mathcal{P}$. Finally, let us consider the worldsheet time translation

$$\delta\mathbb{X}\left(\tau,\sigma\right) \equiv \mathbb{F}\left(\tau\right), \tag{3.44}$$

where $\mathbb{F}$ is any phase-space vector such that $\partial_\sigma \mathbb{F} = 0$. Applying this to the action (3.25), we obtain the boundary term

$$\delta S = \frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\partial_\sigma \mathbb{X}^A\left(\eta - \omega\right)_{AB}\partial_\tau \mathbb{F}^B = \frac{1}{4\pi}\int \mathrm{d}\tau\,\Delta^A\left(\eta - \omega\right)_{AB}\partial_\tau \mathbb{F}^B, \tag{3.45}$$

where $\Delta\left(\tau\right) \equiv \mathbb{X}\left(2\pi,\tau\right) - \mathbb{X}\left(0,\tau\right)$ is the monodromy. The variation vanishes if $\partial_\tau \mathbb{F}$ is in the kernel of $\eta - \omega$, which implies that $\partial_\tau \mathbb{F}$ is null with respect to η, i.e. $\partial_\tau \mathbb{F}\cdot\partial_\tau \mathbb{F} = 0$. Now, under time translation we have

$$\delta\mathbb{S}\left(\tau\right) = \delta\left(\partial_\tau \mathbb{X} - J\partial_\sigma \mathbb{X}\right) = \partial_\tau \mathbb{F}. \tag{3.46}$$

By fixing $\partial_\tau \mathbb{F} \equiv -\mathbb{S}$, which is possible since $\mathbb{S}$ is also a null vector with respect to η, we are able to choose a gauge where $\mathbb{S} = 0$. For the rest of this work we shall work in this gauge.

3.1.5 Implications of Gauge-Fixing $\mathbb{S} = 0$

Now that we have chosen a gauge where $\mathbb{S} = 0$, the Lorentz constraints are satisfied provided that $J^2 = 1$. Thus the action is invariant under worldsheet Lorentz transformations. We have previously found that the energy-momentum tensor is

$$T_{00} = -\frac{1}{2}\partial_1\mathbb{X} \cdot J\partial_1\mathbb{X}, \qquad T_{01} = -\frac{1}{2}\partial_1\mathbb{X} \cdot \partial_1\mathbb{X}, \tag{3.47}$$

$$T_{10} = -\frac{1}{2}\left(2\partial_0\mathbb{X} \cdot J\partial_1\mathbb{X} - \partial_0\mathbb{X} \cdot \partial_0\mathbb{X}\right), \qquad T_{11} = -\frac{1}{2}\partial_1\mathbb{X} \cdot J\partial_1\mathbb{X}. \tag{3.48}$$

The gauge $\mathbb{S} = 0$ means that

$$\mathbb{S} = \partial_0\mathbb{X} - J\partial_1\mathbb{X} = 0. \tag{3.49}$$

Therefore in this gauge we may replace (on-shell) $\partial_0\mathbb{X}$ with $J\partial_1\mathbb{X}$ and vice versa. Note also that $J^{\mathrm{T}}\eta J = \eta$, so $J\mathbb{X} \cdot J\mathbb{X} = \mathbb{X} \cdot \mathbb{X}$. This allows us to write the energy-momentum tensor in a more symmetric form:

$$T_{00} = -\frac{1}{4}\left(\partial_0\mathbb{X} \cdot J\partial_0\mathbb{X} + \partial_1\mathbb{X} \cdot J\partial_1\mathbb{X}\right), \qquad T_{01} = -\frac{1}{2}\partial_0\mathbb{X} \cdot J\partial_1\mathbb{X}, \tag{3.50}$$

$$T_{10} = -\frac{1}{2}\partial_0\mathbb{X} \cdot J\partial_1\mathbb{X}, \qquad T_{11} = -\frac{1}{4}\left(\partial_0\mathbb{X} \cdot J\partial_0\mathbb{X} + \partial_1\mathbb{X} \cdot J\partial_1\mathbb{X}\right). \tag{3.51}$$

Observe that we may write this simply as

$$\begin{aligned} T_{ab} &= -\frac{1}{2}\partial_a\mathbb{X} \cdot J\partial_b\mathbb{X} + \frac{1}{4}\gamma_{ab}\partial_c\mathbb{X} \cdot J\partial^c\mathbb{X} \\ &= -\frac{1}{2}H_{AB}\left(\partial_a\mathbb{X}^A\partial_b\mathbb{X}^B - \frac{1}{2}\gamma_{ab}\partial_c\mathbb{X}^A\partial^c\mathbb{X}^B\right), \end{aligned}$$

where γ_{ab} is the worldsheet Minkowski metric. This is, of course, analogous to the energy-momentum tensor (2.20) for the Polyakov string, with H_{AB} in place of the spacetime metric $h_{\mu\nu}$.

3.2 Chirality In Metastring Theory

3.2.1 Complex (Chiral) Coordinates

In order to calculate OPEs on the metastring worldsheet, we should write everything down in complex coordinates. Recall that the relations between the Lorentzian coordinates and the (Wick-rotated, Euclidean) complex coordinates are

$$\partial_\sigma \mapsto \partial + \bar{\partial}, \qquad \partial_\tau \mapsto \partial - \bar{\partial}. \tag{3.52}$$

Recall also that, by convention, the complex action should have a minus sign relative to the Lorentzian action (3.25), as a Wick rotation is involved, and also that $\mathrm{d}^2\sigma = \mathrm{d}^2z\sqrt{\gamma} = \frac{1}{2}\mathrm{d}^2z$. Putting everything together, we discover that the metastring action in complex coordinates is

$$S = \frac{1}{4\pi}\int \mathrm{d}^2z\left(\partial\mathbb{X}^A\left(H-\omega\right)_{AB}\bar{\partial}\mathbb{X}^B + \frac{1}{2}\partial\mathbb{X}^A\left(H-\eta\right)_{AB}\partial\mathbb{X}^B + \frac{1}{2}\bar{\partial}\mathbb{X}^A\left(H+\eta\right)_{AB}\bar{\partial}\mathbb{X}^B\right), \tag{3.53}$$

where we used the fact that the metrics H and η are symmetric while the symplectic form ω is antisymmetric. We call these coordinates *chiral* since they make manifest the distinction between the holomorphic (left-moving) $\partial\mathbb{X}(z)$ and antiholomorphic (right-moving) $\bar{\partial}\mathbb{X}(\bar{z})$ in metastring theory, which is not present in the standard Polyakov action. The importance of chirality in metastring theory will be further clarified shortly.

A standard calculation gives the energy-momentum tensor in these coordinates:

$$T_{xy} = \begin{pmatrix} -\frac{1}{2}\partial\mathbb{X} \cdot J\partial\mathbb{X} & 0 \\ 0 & -\frac{1}{2}\bar{\partial}\mathbb{X} \cdot J\bar{\partial}\mathbb{X} \end{pmatrix}. \tag{3.54}$$

As a consistency check, recall that the Minkowski metric in complex coordinates is

$$\gamma_{xy} = \frac{\partial\sigma^a}{\partial z^x}\frac{\partial\sigma^b}{\partial z^y}\gamma_{ab} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.55}$$

Thus

$$T^x_{\ x} \equiv \gamma^{xy}T_{xy} = 2\left(T_{z\bar{z}} + T_{z\bar{z}}\right) = 0, \tag{3.56}$$

and the energy-momentum tensor is traceless as required. We define a holomorphic (left-moving) component $T(z)$ and an antiholomorphic (right-moving) component $\bar{T}(\bar{z})$:

$$T(z) \equiv T_{zz} = -\frac{1}{2}\partial\mathbb{X}\cdot J\partial\mathbb{X}, \qquad \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}} = -\frac{1}{2}\bar{\partial}\mathbb{X}\cdot J\bar{\partial}\mathbb{X}. \tag{3.57}$$

The chiral action (3.53) has the appealing property that it is symmetric in the worldsheet coordinates, unlike (3.25). The second (holomorphic, left-moving) term differs from the third (antiholomorphic, right-moving) term by the sign of the P-metric η.

3.2.2 The Chiral Structure J and Chiral Projectors

Explicitly, the chiral structure $J \equiv \eta^{-1}H$ is[9]

$$J = \eta^{-1}H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} = \begin{pmatrix} 0 & h^{-1} \\ h & 0 \end{pmatrix}. \tag{3.58}$$

It's compatible with η and H:

$$J^{\mathrm{T}}\eta = H = \eta J \quad\Longrightarrow\quad J^{\mathrm{T}}\eta J = \eta, \tag{3.59}$$

$$J^{\mathrm{T}}H = H\eta^{-1}H = HJ \quad\Longrightarrow\quad J^{\mathrm{T}}HJ = H. \tag{3.60}$$

We use the chiral structure to define

$$P_\pm \equiv \frac{1}{2}\left(1 \pm J\right) = \frac{1}{2}\eta^{-1}\left(\eta \pm H\right), \tag{3.61}$$

such that

$$P_\pm = \frac{1}{2}\left(1 \pm J\right) = \frac{1}{2}\begin{pmatrix} 1 & \pm h^{-1} \\ \pm h & 1 \end{pmatrix}. \tag{3.62}$$

Acting on a phase-space vector $\mathbb{X} = (X, Y)$, we get

$$P_\pm\mathbb{X} = \frac{1}{2}\begin{pmatrix} 1 & \pm h^{-1} \\ \pm h & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} X \pm Y \\ Y \pm X \end{pmatrix}. \tag{3.63}$$

We interpret $P_\pm\mathbb{X}$ as the *chiral components* of $\mathbb{X}$ and $P_\pm$ as *chiral projectors*; they satisfy

$$\mathbb{X} = P_+\mathbb{X} + P_-\mathbb{X}, \qquad P_+P_-\mathbb{X} = P_-P_+\mathbb{X} = 0, \qquad P_\pm^2 = P_\pm. \tag{3.64}$$

Observe that, since $J^2 = 1$, we have

$$JP_\pm\mathbb{X} = \pm P_\pm\mathbb{X}, \tag{3.65}$$

so the $P_\pm$ project on the eigenspaces ±1 of J. This provides us with a notion of target space (i.e. phase space) chirality, determined by the J eigenvalue.

Let us find a relation between worldsheet chirality, as determined by ∂ and $\bar{\partial}$, and target space chirality, as determined by the $J = \pm1$ eigenspaces. In complex coordinates we have

$$\mathbb{S} \equiv \partial_\tau\mathbb{X} - J\partial_\sigma\mathbb{X} = (1 - J)\,\partial\mathbb{X} - (1 + J)\,\bar{\partial}\mathbb{X} = 2P_-\partial\mathbb{X} - 2P_+\bar{\partial}\mathbb{X}. \tag{3.66}$$

Then setting $\mathbb{S} = 0$ means that

$$P_-\partial\mathbb{X} = P_+\bar{\partial}\mathbb{X}. \tag{3.67}$$

[9] Recall that we are assuming a background with vanishing B-field.

Now, we have

$$\partial_\sigma \mathbb{X} = \partial\mathbb{X} + \bar{\partial}\mathbb{X}, \qquad \partial_\tau \mathbb{X} = \partial\mathbb{X} - \bar{\partial}\mathbb{X}. \tag{3.68}$$

Taking the projections of these equations and using $P_-\partial\mathbb{X} = P_+\bar{\partial}\mathbb{X}$ gives

$$\partial\mathbb{X} = P_+\partial_\sigma\mathbb{X} = P_+J\partial_\tau\mathbb{X}, \qquad \bar{\partial}\mathbb{X} = P_-\partial_\sigma\mathbb{X} = P_-J\partial_\tau\mathbb{X}. \tag{3.69}$$

This tells us that the holomorphic (left-moving) fields $\partial\mathbb{X}$ and antiholomorphic (right-moving) fields $\bar{\partial}\mathbb{X}$ on the worldsheet are also the projections of $\partial_\sigma\mathbb{X}$ (or $J\partial_\tau\mathbb{X}$) on the $J = +1$ and $J = -1$ eigenspaces, respectively.

3.2.3 Chiral Notation

Given an arbitrary phase-space vector $\mathbb{V}$, we can separate it into $J = \pm 1$ components:

$$\mathbb{V} = P_+\mathbb{V} + P_-\mathbb{V}, \tag{3.70}$$

or in index notation

$$\mathbb{V}^A = (P_+)^A{}_B\,\mathbb{V}^B + (P_-)^A{}_B\,\mathbb{V}^B. \tag{3.71}$$

We therefore adopt the notation

$$\mathbb{V}_\pm \equiv P_\pm\mathbb{V}, \qquad \mathbb{V} \equiv \mathbb{V}_+ + \mathbb{V}_-. \tag{3.72}$$

With this notation, we have

$$(\partial_\sigma\mathbb{X})_+ = \partial\mathbb{X}, \qquad (\partial_\sigma\mathbb{X})_- = \bar{\partial}\mathbb{X}. \tag{3.73}$$

Therefore we adopt the suggestive notation

$$\partial_+\mathbb{X} \equiv \partial\mathbb{X}, \qquad \partial_-\mathbb{X} \equiv \bar{\partial}\mathbb{X}, \tag{3.74}$$

such that

$$P_+\partial_+\mathbb{X} = \partial_+\mathbb{X}, \qquad P_-\partial_-\mathbb{X} = \partial_-\mathbb{X}, \qquad P_+\partial_-\mathbb{X} = P_-\partial_+\mathbb{X} = 0. \tag{3.75}$$

Similarly, given an arbitrary rank-2 tensor V, we can separate it into four $J = \pm 1$ components:

$$V = V_{++} + V_{+-} + V_{-+} + V_{--}, \tag{3.76}$$

where, for $X, Y \in \{+, -\}$,

$$(V_{XY})_{AB} \equiv (P_X)^C{}_A\,V_{CD}\,(P_Y)^D{}_B \quad \Longrightarrow \quad V_{XY} \equiv P_X^{\mathrm{T}} V P_Y. \tag{3.77}$$

If the tensor has upper indices, then this would be instead

$$(V_{XY})^{AB} \equiv (P_X)^A{}_C\,V^{CD}\,(P_Y)^B{}_D \quad \Longrightarrow \quad V_{XY} \equiv P_X V P_Y^{\mathrm{T}}. \tag{3.78}$$

Similarly, if the tensor has one upper and one lower index, the we would have

$$(V_{XY})^A{}_B \equiv (P_X)^A{}_C\,V^C_D\,(P_Y)^D{}_B \quad \Longrightarrow \quad V_{XY} \equiv P_X V P_Y. \tag{3.79}$$

For brevity we will sometimes not write the transpose symbol explicitly; it should then be understood from context.

3.2.4 Symmetry Properties of Projections

Any rank-2 tensor may be separated into symmetric and antisymmetric components:

$$V_{AB}=\left(\frac{V_{AB}+V_{BA}}{2}\right)+\left(\frac{V_{AB}-V_{BA}}{2}\right)\equiv V_{(AB)}+V_{[AB]}\equiv S_{AB}+A_{AB}. \tag{3.80}$$

Let us project the symmetric and antisymmetric parts individually:

$$(S_{XY})_{AB}=(P_X)^C{}_A\,S_{CD}\,(P_Y)^D{}_B\,,\qquad (A_{XY})_{AB}=(P_X)^C{}_A\,A_{CD}\,(P_Y)^D{}_B\,. \tag{3.81}$$

It's easy to see that

$$(S_{XY})_{AB}=(S_{YX})_{BA}\,,\qquad (A_{XY})_{AB}=-\,(A_{YX})_{BA}\,. \tag{3.82}$$

In particular, if $X=Y$ then the symmetric (antisymmetric) part of the projection is the projection of the symmetric (antisymmetric) part:

$$(V_{XX})_{(AB)}=(S_{XX})_{AB}\,,\qquad (V_{XX})_{[AB]}=(A_{XX})_{AB}\,. \tag{3.83}$$

In other words, a diagonal ($\pm\pm$) projection preserves the symmetry or antisymmetry of the projected tensor. For $X\neq Y$ we have

$$(V_{XY})_{AB}=(S_{XY})_{AB}+(A_{XY})_{AB}\,,\qquad (V_{YX})_{BA}=(S_{XY})_{AB}-(A_{XY})_{AB}\,. \tag{3.84}$$

This means that for any given tensor V, we don't need to consider both off-diagonal projections V_{+-} and V_{-+} individually; it is sufficient to consider only the $+-$ projection, while projecting the symmetric and antisymmetric parts of V individually.

3.2.5 Important Projections

Let us find the projections of various tensors of interest. For the projectors $P_\pm$ themselves we have the obvious result

$$(P_\pm)_{\pm\pm}=P_\pm,\qquad (P_\pm)_{\mp\mp}=(P_\pm)_{\pm\mp}=(P_\pm)_{\mp\pm}=0. \tag{3.85}$$

For the chiral structure J we have, using $J^2=1$,

$$J_{XY}=\frac{1}{4}\left((X+Y)+(1+XY)\,J\right). \tag{3.86}$$

Thus

$$J_{\pm\pm}=\pm P_\pm,\qquad J_{\pm\mp}=0. \tag{3.87}$$

For the metrics H and η we shall use the compatibility relations

$$J^{\mathrm{T}}\eta=H=\eta J\quad\Longrightarrow\quad J^{\mathrm{T}}\eta J=\eta, \tag{3.88}$$

$$J^{\mathrm{T}}H=\eta=HJ\quad\Longrightarrow\quad J^{\mathrm{T}}HJ=H. \tag{3.89}$$

We see that

$$\begin{aligned}H_{XY}&=\frac{1}{4}\left(1+XJ^{\mathrm{T}}\right)H\,(1+YJ)\\&=\frac{1}{4}\left((1+XY)\,H+(X+Y)\,\eta\right),\end{aligned}$$

$$\begin{aligned}\eta_{XY}&=\frac{1}{4}\left(1+XJ^{\mathrm{T}}\right)\eta\,(1+YJ)\\&=\frac{1}{4}\left((1+XY)\,\eta+(X+Y)\,H\right).\end{aligned}$$

Therefore

$$H_{\pm\pm} = \pm\eta_{\pm\pm} = \frac{1}{2}\left(H \pm \eta\right), \qquad H_{\pm\mp} = \eta_{\pm\mp} = 0. \tag{3.90}$$

J is also compatible with the inverse metrics:

$$JH^{-1} = \eta^{-1} = H^{-1}J^{\mathrm{T}} \quad \Longrightarrow \quad JH^{-1}J^{\mathrm{T}} = H^{-1}, \tag{3.91}$$

$$J\eta^{-1} = H^{-1} = \eta^{-1}J^{\mathrm{T}} \quad \Longrightarrow \quad J\eta^{-1}J^{\mathrm{T}} = \eta^{-1}. \tag{3.92}$$

Hence we have the corresponding relations

$$H^{-1}_{\pm\pm} = \pm\eta^{-1}_{\pm\pm} = \frac{1}{2}\left(H^{-1} \pm \eta^{-1}\right), \qquad H^{-1}_{\pm\mp} = \eta^{-1}_{\pm\mp} = 0. \tag{3.93}$$

Here we are abusing notation slightly. It must be stressed that, for example, H^{-1}_{++} means $\left(H^{-1}\right)_{++}$, the $++$ projection of H^{-1}, and not the inverse of H_{++}; projected matrices are not invertible.

We conclude that the metrics H and η are, in some way, "equivalent" up to a sign when projected on the chiral spaces. Thus, when dealing with projections, it will usually not matter much whether we use H or η. For example, the components of the energy-momentum tensor may be written in various equivalent ways:

$$T(z) = -\frac{1}{2}\partial\mathbb{X}\cdot J\partial\mathbb{X} = -\frac{1}{2}\partial\mathbb{X}H\partial\mathbb{X} = -\frac{1}{2}\partial\mathbb{X}\eta\partial\mathbb{X} = -\frac{1}{4}\partial_\sigma\mathbb{X}\left(H+\eta\right)\partial_\sigma\mathbb{X}, \tag{3.94}$$

$$\bar{T}(\bar{z}) = -\frac{1}{2}\bar{\partial}\mathbb{X}\cdot J\bar{\partial}\mathbb{X} = -\frac{1}{2}\bar{\partial}\mathbb{X}H\bar{\partial}\mathbb{X} = +\frac{1}{2}\bar{\partial}\mathbb{X}\eta\bar{\partial}\mathbb{X} = -\frac{1}{4}\partial_\sigma\mathbb{X}\left(H-\eta\right)\partial_\sigma\mathbb{X}. \tag{3.95}$$

3.3 Calculating OPEs

3.3.1 The Propagator

We now repeat the calculation done in section (2.4) for the chiral metastring action (3.53):

$$S = \frac{1}{4\pi}\int \mathrm{d}^2 z\left(\partial\mathbb{X}^A\left(H-\omega\right)_{AB}\bar{\partial}\mathbb{X}^B + \frac{1}{2}\partial\mathbb{X}^A\left(H-\eta\right)_{AB}\partial\mathbb{X}^B + \frac{1}{2}\bar{\partial}\mathbb{X}^A\left(H+\eta\right)_{AB}\bar{\partial}\mathbb{X}^B\right). \tag{3.96}$$

As before, we use the fact that the path integral of a total functional derivative vanishes:

$$\begin{aligned}
0 &= \int \mathcal{D}\mathbb{X}\frac{\delta}{\delta\mathbb{X}^A(z,\bar{z})}\left(\mathrm{e}^{-S}\,\mathbb{X}^B(w,\bar{w})\right)\\
&= \int \mathcal{D}\mathbb{X}\,\mathrm{e}^{-S}\Bigg(\delta^B_A\delta\left(z-w,\bar{z}-\bar{w}\right)+\\
&\qquad +\frac{1}{4\pi}\left(2H_{AC}\partial\bar{\partial}+\left(H-\eta\right)_{AC}\partial\partial+\left(H+\eta\right)_{AC}\bar{\partial}\bar{\partial}\right)\mathbb{X}^C(z,\bar{z})\,\mathbb{X}^B(w,\bar{w})\Bigg)\\
&= \delta^B_A\left\langle\delta\left(z-w,\bar{z}-\bar{w}\right)\right\rangle + \frac{1}{4\pi}\left\langle\left(2H_{AC}\partial\bar{\partial}+\left(H-\eta\right)_{AC}\partial\partial+\left(H+\eta\right)_{AC}\bar{\partial}\bar{\partial}\right)\mathbb{X}^C(z,\bar{z})\,\mathbb{X}^B(w,\bar{w})\right\rangle.
\end{aligned}$$

Hence the following operator equation holds:

$$\left(H_{AC}\partial\bar{\partial}+\frac{1}{2}\left(H-\eta\right)_{AC}\partial\partial+\frac{1}{2}\left(H+\eta\right)_{AC}\bar{\partial}\bar{\partial}\right)\mathbb{X}^C(z,\bar{z})\,\mathbb{X}^B(w,\bar{w}) = -2\pi\delta^B_A\delta\left(z-w,\bar{z}-\bar{w}\right). \tag{3.97}$$

To solve this equation, first recall the definition of the chiral structure and projectors:

$$J \equiv \eta^{-1}H, \qquad P_\pm \equiv \frac{1}{2}\left(1\pm J\right) = \frac{1}{2}\eta^{-1}\left(\eta\pm H\right). \tag{3.98}$$

Multiplying equation (3.97) by η^{DA}, we thus get:

$$\left(J\partial\bar{\partial}-P_-\partial\partial+P_+\bar{\partial}\bar{\partial}\right)^D{}_C\,\mathbb{X}^C(z,\bar{z})\,\mathbb{X}^B(w,\bar{w}) = -2\pi\eta^{BD}\delta\left(z-w,\bar{z}-\bar{w}\right). \tag{3.99}$$

Next, in section 2.4 we found that for $z \to w$:

$$\partial\bar{\partial} \ln |z-w|^2 = \partial \frac{1}{\bar{z}-\bar{w}} = \bar{\partial} \frac{1}{z-w} = 2\pi\delta\left(z-w, \bar{z}-\bar{w}\right). \tag{3.100}$$

Finally, recall the result of section 3.2.5:

$$H^{-1}_{\pm\pm} = \pm\eta^{-1}_{\pm\pm} = \frac{1}{2}\left(H^{-1} \pm \eta^{-1}\right). \tag{3.101}$$

Using these results, let us now define

$$G\left(z-w, \bar{z}-\bar{w}\right) \equiv -H^{-1}_{++} \ln\left(z-w\right) - H^{-1}_{--} \ln\left(\bar{z}-\bar{w}\right). \tag{3.102}$$

Then we have:

$$\begin{aligned} J\partial\bar{\partial}G\left(z-w, \bar{z}-\bar{w}\right) &= -J\left(H^{-1}_{++}\bar{\partial}\partial \ln\left(z-w\right) + H^{-1}_{--}\partial\bar{\partial}\ln\left(\bar{z}-\bar{w}\right)\right) \\ &= -J\left(H^{-1}_{++} + H^{-1}_{--}\right) 2\pi\delta\left(z-w, \bar{z}-\bar{w}\right) \\ &= -\left(H^{-1}_{++} - H^{-1}_{--}\right) 2\pi\delta\left(z-w, \bar{z}-\bar{w}\right) \\ &= -2\pi\eta^{-1}\delta\left(z-w, \bar{z}-\bar{w}\right), \end{aligned}$$

where we used $JH^{-1}_{\pm\pm} = \pm H^{-1}_{\pm\pm}$ and $H^{-1}_{++} - H^{-1}_{--} = \eta^{-1}$. We also have:

$$\begin{aligned} P_{-}\partial\partial G\left(z-w, \bar{z}-\bar{w}\right) &= -P_{-}\left(H^{-1}_{++}\partial\partial \ln\left(z-w\right) + H^{-1}_{--}\partial\partial\ln\left(\bar{z}-\bar{w}\right)\right) \\ &= P_{-}H^{-1}_{++}\frac{1}{\left(z-w\right)^2} = 0, \end{aligned}$$

$$\begin{aligned} P_{+}\bar{\partial}\bar{\partial}G\left(z-w, \bar{z}-\bar{w}\right) &= -P_{+}\left(H^{-1}_{++}\bar{\partial}\bar{\partial} \ln\left(z-w\right) + H^{-1}_{--}\bar{\partial}\bar{\partial}\ln\left(\bar{z}-\bar{w}\right)\right) \\ &= P_{+}H^{-1}_{--}\frac{1}{\left(\bar{z}-\bar{w}\right)^2} = 0. \end{aligned}$$

Therefore, G is a solution to equation (3.97), and thus the propagator for the worldsheet fields in metastring theory is[10]:

$$\mathbb{X}^A\left(z, \bar{z}\right)\mathbb{X}^B\left(w, \bar{w}\right) = -H^{AB}_{++}\ln\left(z-w\right) - H^{AB}_{--}\ln\left(\bar{z}-\bar{w}\right). \tag{3.103}$$

To see the relation to the usual string theory propagator (2.31), let us write the projected metrics explicitly:

$$\begin{aligned} \mathbb{X}^A\left(z, \bar{z}\right)\mathbb{X}^B\left(w, \bar{w}\right) &= -\frac{1}{2}\left(H^{AB} + \eta^{AB}\right)\ln\left(z-w\right) - \frac{1}{2}\left(H^{AB} - \eta^{AB}\right)\ln\left(\bar{z}-\bar{w}\right) \\ &= -\frac{1}{2}H^{AB}\left(\ln\left(z-w\right) + \ln\left(\bar{z}-\bar{w}\right)\right) - \frac{1}{2}\eta^{AB}\left(\ln\left(z-w\right) - \ln\left(\bar{z}-\bar{w}\right)\right) \\ &= -\frac{1}{2}H^{AB}\ln|z-w|^2 - \frac{1}{2}\eta^{AB}\ln\frac{z-w}{\bar{z}-\bar{w}}. \end{aligned}$$

The first term, proportional to H, is equivalent to the usual propagator (2.31). However, the second term, proportional to η, is a new feature of metastring theory.

3.3.2 Wick's Theorem and Contractions

We shall take $w \to 0$ in the OPEs for brevity. Recall that the OPE of two normal-ordered operators $\mathcal{A}$ and $\mathcal{B}$ is given by

$$:\mathcal{A}::\mathcal{B}:=:\mathcal{A}\mathcal{B}: + \sum \text{contractions}, \tag{3.104}$$

[10] As we have stressed previously, the notation $H^{AB}_{\pm\pm}$ should be understood as the the $\pm\pm$ projection of H^{AB}, that is, of H^{-1}.

where now the sum is over contractions of pairs of fields $\mathbb{X}$, one from $\mathcal{A}$ and one from $\mathcal{B}$, replacing each pair with the propagator (3.103):

$$\langle \mathbb{X}^A (z, \bar{z}) \mathbb{X}^B (0,0) \rangle = -H^{AB}_{++} \ln z - H^{AB}_{--} \ln \bar{z}. \tag{3.105}$$

By taking derivatives of this expression we may derive additional contractions[11]:

$$\langle \partial \mathbb{X}^A (z) \partial \mathbb{X}^B (0) \rangle = -H^{AB}_{++} \frac{1}{z^2}, \tag{3.106}$$

$$\langle \bar{\partial} \mathbb{X}^A (\bar{z}) \bar{\partial} \mathbb{X}^B (0) \rangle = -H^{AB}_{--} \frac{1}{\bar{z}^2}, \tag{3.107}$$

$$\langle \partial \mathbb{X}^A (z) \bar{\partial} \mathbb{X}^B (0) \rangle = 0. \tag{3.108}$$

We would also like to be able to contract $\partial \mathbb{X}$ or $\bar{\partial} \mathbb{X}$ with an exponential of the form $\mathrm{e}^{\mathrm{i}\,\mathbb{P} \cdot \mathbb{X}}$, where $\mathbb{P}$ is a phase-space covector. We have:

$$\langle \partial \mathbb{X}^A (z) \mathbb{X}^B (0,0) \rangle = -H^{AB}_{++} \frac{1}{z}, \tag{3.109}$$

$$\langle \bar{\partial} \mathbb{X}^A (\bar{z}) \mathbb{X}^B (0,0) \rangle = -H^{AB}_{--} \frac{1}{\bar{z}}. \tag{3.110}$$

The calculation is completely analogous to the one performed in section 2.5:

$$\langle \partial \mathbb{X}^A (z) \, \mathrm{e}^{\mathrm{i}\,\mathbb{P} \cdot \mathbb{X}(0,0)} \rangle = -\,\mathrm{i}\, H^{AB}_{++} \mathbb{P}_B \frac{1}{z} \, \mathrm{e}^{\mathrm{i}\,\mathbb{P} \cdot \mathbb{X}(0,0)}, \tag{3.111}$$

$$\langle \bar{\partial} \mathbb{X}^A (\bar{z}) \, \mathrm{e}^{\mathrm{i}\,\mathbb{P} \cdot \mathbb{X}(0,0)} \rangle = -\,\mathrm{i}\, H^{AB}_{--} \mathbb{P}_B \frac{1}{\bar{z}} \, \mathrm{e}^{\mathrm{i}\,\mathbb{P} \cdot \mathbb{X}(0,0)}. \tag{3.112}$$

To make the calculations more efficient, we employ a chiral notation where $X, Y \in \{+, -\}$ and for any operator $\mathcal{O}$ we have

$$\mathcal{O}(X) \equiv \begin{cases} \mathcal{O}(z) & X = +, \\ \mathcal{O}(\bar{z}) & X = -. \end{cases} \tag{3.113}$$

Using $H_{+-} = H_{-+} = 0$, we can now summarize all the different contractions neatly as follows:

$$\langle \partial_X \mathbb{X}^A (X) \partial_Y \mathbb{X}^B (0) \rangle = -H^{AB}_{XY} \frac{1}{X^2}, \tag{3.114}$$

$$\langle \partial_X \mathbb{X}^A (X) \, \mathrm{e}^{\mathrm{i}\,\mathbb{P} \cdot \mathbb{X}(0,0)} \rangle = -\,\mathrm{i}\, H^{AB}_{XX} \mathbb{P}_B \frac{1}{X} \, \mathrm{e}^{\mathrm{i}\,\mathbb{P} \cdot \mathbb{X}(0,0)}. \tag{3.115}$$

3.3.3 The TT OPE

With the tools of section 3.3.2, we now proceed to calculate the OPE of the holomorphic and antiholomorphic components of the energy-momentum tensor for the metastring with themselves. In chiral notation the energy-momentum tensor is expressed as[12]

$$T_X (X) \equiv -\frac{1}{2} : \partial_X \mathbb{X}^A H^{XX}_{AB} \partial_X \mathbb{X}^B : . \tag{3.116}$$

We would like to calculate the OPE of

$$T_X (X) T_Y (0) = \frac{1}{4} H^{XX}_{AB} H^{YY}_{CD} : \partial_X \mathbb{X}^A (X) \partial_X \mathbb{X}^B (X) :: \partial_Y \mathbb{X}^C (0) \partial \mathbb{X}^D (0) : . \tag{3.117}$$

[11] Note that in these expressions the first derivative is with respect to z or $\bar{z}$, but the second is with respect to w or $\bar{w}$, which contributes an extra minus sign.

[12] This notation is not strictly necessary for this relatively simple calculation. However, we introduce it here in order to familiarize the reader with it. It will be indispensable in Appendix B.

To form contractions we may contract a single pair, which is possible in 4 ways: $\langle AC\rangle, \langle AD\rangle, \langle BC\rangle, \langle BD\rangle$, or two pairs, which is possible in 2 ways: $\langle AC\rangle\langle BD\rangle$ and $\langle AD\rangle\langle BC\rangle$. Thus we get

$$\begin{aligned} T_X(X)\,T_Y(0) &\sim 2\cdot\frac{1}{4}H^{XX}_{AB}H^{YY}_{CD}\left\langle\partial_X\mathbb{X}^A(X)\,\partial_Y\mathbb{X}^C(0)\right\rangle\left\langle\partial_X\mathbb{X}^B(X)\,\partial_Y\mathbb{X}^D(0)\right\rangle+\\ &\quad+4\cdot\frac{1}{4}H^{XX}_{AB}H^{YY}_{CD}\left\langle\partial_X\mathbb{X}^A(X)\,\partial_Y\mathbb{X}^C(0)\right\rangle:\partial_X\mathbb{X}^B(X)\,\partial_Y\mathbb{X}^D(0):\\ &=\frac{1/2}{X^4}H^{XX}_{AB}H^{YY}_{CD}H^{AC}_{XY}H^{BD}_{XY}-\frac{1}{X^2}H^{XX}_{AB}H^{YY}_{CD}H^{AC}_{XY}:\partial_X\mathbb{X}^B(X)\,\partial_Y\mathbb{X}^D(0):.\end{aligned}$$

To simplify, we note that

$$H^{AB}H_{BC}=\left(H^{-1}H\right)^A{}_C=\delta^A_C. \tag{3.118}$$

If we project both sides onto $\pm\pm$ we get the corresponding projector in the right-hand side:

$$H^{AB}_{\pm\pm}H^{\pm\pm}_{BC}=\left(P_\pm\right)^A{}_C. \tag{3.119}$$

All other possible combinations of projections of the H's are easily seen to vanish. In chiral notation, this may be written as

$$H^{AB}_{WX}H^{YZ}_{BC}=\left(P_W\right)^A{}_C\,\delta_{WX}\delta_{XY}\delta_{YZ}=\begin{cases}\left(P_W\right)^A{}_C & W=X=Y=Z,\\ 0 & \text{otherwise.}\end{cases} \tag{3.120}$$

Observe also that

$$H^{XX}_{AB}H^{AB}_{XX}=\operatorname{tr}P_X=D. \tag{3.121}$$

The products involving projections of H in our OPE may now be calculated:

$$T_X(X)\,T_Y(0)\sim\delta_{XY}\left(\frac{D/2}{X^4}-\frac{1}{X^2}H^{XX}_{AB}:\partial_X\mathbb{X}^A(X)\,\partial_X\mathbb{X}^B(0):\right). \tag{3.122}$$

Finally, we expand $\partial_X\mathbb{X}(X)\approx\partial_X\mathbb{X}(0)+X\partial_X\partial_X\mathbb{X}(0)$, discard terms regular as $X\to 0$, and substitute

$$T_X=-\frac{1}{2}:\partial_X\mathbb{X}^A H^{XX}_{AB}\partial_X\mathbb{X}^B:,\qquad \partial_X T_X=-:\partial_X\partial_X\mathbb{X}^A H^{XX}_{AB}\partial_X\mathbb{X}^B:, \tag{3.123}$$

obtaining

$$T_X(X)\,T_Y(0)\sim\delta_{XY}\left(\frac{D/2}{X^4}+\frac{2T_X(0)}{X^2}+\frac{\partial_X T_X}{X}\right). \tag{3.124}$$

Note that the central charge is $c=D$, just as it was in the original string theory. This was to be expected, seeing that we have not actually introduced any new degrees of freedom. In normal notation, this equation is expressed as follows:

$$\begin{aligned} T(z)\,T(0)&\sim\frac{c/2}{z^4}+\frac{2T(0)}{z^2}+\frac{\partial T(0)}{z},\\ \bar{T}(\bar{z})\,\bar{T}(0)&\sim\frac{\bar{c}/2}{\bar{z}^4}+\frac{2\bar{T}(0)}{\bar{z}^2}+\frac{\bar{\partial}\bar{T}(0)}{\bar{z}},\\ T(z)\,\bar{T}(0)&\sim 0,\end{aligned} \tag{3.125}$$

where $c=\bar{c}=D$. This entire calculation may be seen as a consistency check; that is, it merely shows that our choices of various factors and signs were consistent.

3.4 Perturbations of the Metageometry

3.4.1 Projections of the Perturbations

We would like to perturb the metrics $H\mapsto H+\delta H$ and $\eta\mapsto\eta+\delta\eta$ and find the equations of motion for the perturbations. First, let us derive an important relation between the chiral projections of the metrics.

From the relation $H = \eta J$ we have

$$\delta H = \delta\eta J + \eta\delta J. \tag{3.126}$$

Multiplying this equation by J from the right, we get

$$\delta H J = \delta\eta + \eta\delta J J. \tag{3.127}$$

Note that from $J^2 = 1$ we have

$$0 = \delta\left(J^2\right) = \delta J J + J\delta J, \tag{3.128}$$

so we can write this using $\eta J = H$ as

$$\delta H J = \delta\eta - H\delta J. \tag{3.129}$$

Alternatively, multiplying equation (3.126) by J^{T} from the left and using $J^{\mathrm{T}}\eta = H$ we get

$$J^{\mathrm{T}}\delta H = J^{\mathrm{T}}\delta\eta J + H\delta J. \tag{3.130}$$

Adding both equations, we obtain

$$J^{\mathrm{T}}\delta H + \delta H J = J^{\mathrm{T}}\delta\eta J + \delta\eta. \tag{3.131}$$

In this way, we have gotten rid of δJ. Now we can project this equation on XY:[13]

$$(X+Y)\,\delta H_{XY} = (1+XY)\,\delta\eta_{XY}. \tag{3.132}$$

From this we get the relation

$$\delta H_{\pm\pm} = \pm\delta\eta_{\pm\pm}. \tag{3.133}$$

This relation is rather remarkable, as it tells us that δH and $\delta\eta$ are not independent. Their diagonal projections are in fact the same, up to a sign. Only the off-diagonal projections, $\delta H_{\pm\mp}$ and $\delta\eta_{\pm\mp}$, may in general be independent of one another.

Moreover, recall from section 3.2.4 that, since δH and $\delta\eta$ are symmetric, their $-+$ projections are simply the transpose of the $+-$ projections. Hence, although it may seem that there are 8 different projections δH_{XY} and $\delta\eta_{XY}$ for $X, Y \in \{+,-\}$, there are really only 4 independent ones. We will take these to be δH_{++}, δH_{--}, δH_{+-} and $\delta\eta_{+-}$.

3.4.2 The Perturbation of the Energy-Momentum Tensor

The Lorentzian metastring action (3.25) is of the form

$$S \sim \int \mathrm{d}^2\sigma\left(\partial_\tau\mathbb{X}^A\left(\eta+\omega\right)_{AB}\partial_\sigma\mathbb{X}^B - \partial_\sigma\mathbb{X}^A H_{AB}\partial_\sigma\mathbb{X}^B\right). \tag{3.134}$$

Thus the conjugate momentum to $\mathbb{X}$ is

$$\mathbb{P}_A \equiv \frac{\delta S}{\delta\left(\partial_\tau\mathbb{X}^A\right)} = \eta_{AB}\partial_\sigma\mathbb{X}^B. \tag{3.135}$$

The symplectic form does not contribute here, since it is a boundary term:

$$\begin{aligned}-2\omega_{AB}\partial_\tau\mathbb{X}^A\partial_\sigma\mathbb{X}^B\,\mathrm{d}^2\sigma &= \omega_{AB}\left(\partial_\tau\mathbb{X}^A\partial_\sigma\mathbb{X}^B - \partial_\sigma\mathbb{X}^A\partial_\tau\mathbb{X}^B\right)\mathrm{d}\tau\wedge\mathrm{d}\sigma\\ &= \omega_{AB}\,\mathrm{d}\mathbb{X}^A\wedge\mathrm{d}\mathbb{X}^B\\ &= \mathrm{d}\left(\omega_{AB}\mathbb{X}^A\,\mathrm{d}\mathbb{X}^B\right).\end{aligned}$$

Of course, this is merely a heuristic treatment; a more rigorous one may be found in [2], chapter 4.

[13] For brevity we write $\delta H_{XY} \equiv (\delta H)_{XY}$ and so on. δH_{XY} is the projection of the perturbation δH, not the perturbation of the projection of H.

To avoid writing everything twice, we again employ chiral notation: $\partial_+ \equiv \partial$, $\partial_- \equiv \bar{\partial}$ and $T_+ \equiv T$, $T_- \equiv \bar{T}$. The components of the energy-momentum tensor may be expressed as:

$$T_\pm = -\frac{1}{4}\partial_\sigma \mathbb{X}^A \left(H_{AB} \pm \eta_{AB}\right) \partial_\sigma \mathbb{X}^B. \tag{3.136}$$

The holomorphic and antiholomorphic derivatives are given in terms of $\mathbb{P}$ by

$$\partial_\pm \mathbb{X}^A = \frac{1}{2}\left(\partial_\sigma \mathbb{X}^A \pm \partial_\tau \mathbb{X}^A\right) = \frac{1}{2}\left(\eta^{AB}\mathbb{P}_B \pm \partial_\tau \mathbb{X}^A\right), \tag{3.137}$$

which may be inverted to find

$$\mathbb{P}_A = \eta_{AB}\left(\partial_+ \mathbb{X}^B + \partial_- \mathbb{X}^B\right), \qquad \partial_\tau \mathbb{X}^A = \partial_+ \mathbb{X}^A - \partial_- \mathbb{X}^A. \tag{3.138}$$

Thus we have

$$\begin{aligned}
\delta\left(\partial_\sigma \mathbb{X}^B\right) &= \delta\left(\eta^{BC}\mathbb{P}_C\right) \\
&= \delta\eta^{BC}\mathbb{P}_C \\
&= \delta\eta^{BC}\eta_{CD}\left(\partial_+ \mathbb{X}^D + \partial_- \mathbb{X}^D\right) \\
&= -\eta^{BC}\delta\eta_{CD}\left(\partial_+ \mathbb{X}^D + \partial_- \mathbb{X}^D\right).
\end{aligned}$$

Now, the perturbation of the energy-momentum tensor is given by

$$\delta T_\pm = -\frac{1}{4}\partial_\sigma \mathbb{X}^A \left(\delta H_{AB} \pm \delta\eta_{AB}\right) \partial_\sigma \mathbb{X}^B - \frac{1}{2}\partial_\sigma \mathbb{X}^A \left(H_{AB} \pm \eta_{AB}\right) \delta\left(\partial_\sigma \mathbb{X}^B\right). \tag{3.139}$$

For the first term we use the fact that $\delta H_{\pm\pm} = \pm\delta\eta_{\pm\pm}$, so

$$\partial_\pm \mathbb{X}\left(\delta H \mp \delta\eta\right)\partial_\pm \mathbb{X} = 0, \qquad \partial_\pm \mathbb{X}\left(\delta H \pm \delta\eta\right)\partial_\pm \mathbb{X} = 2\partial_\pm \mathbb{X}\delta H \partial_\pm \mathbb{X}, \tag{3.140}$$

which gives

$$\begin{aligned}
-\frac{1}{4}\partial_\sigma \mathbb{X}^A \left(\delta H_{AB} \pm \delta\eta_{AB}\right)\partial_\sigma \mathbb{X}^B &= -\frac{1}{4}\left(\partial_+ \mathbb{X}^A + \partial_- \mathbb{X}^A\right)\left(\delta H_{AB} \pm \delta\eta_{AB}\right)\left(\partial_+ \mathbb{X}^B + \partial_- \mathbb{X}^B\right) \\
&= -\frac{1}{2}\partial_\pm \mathbb{X}^A \delta H_{AB}\partial_\pm \mathbb{X}^B - \frac{1}{2}\partial_+ \mathbb{X}^A\left(\delta H_{AB} \pm \delta\eta_{AB}\right)\partial_- \mathbb{X}^B.
\end{aligned}$$

For the second term we have

$$\begin{aligned}
-\frac{1}{2}\partial_\sigma \mathbb{X}^A \left(H_{AB} \pm \eta_{AB}\right)\delta\left(\partial_\sigma \mathbb{X}^B\right) &= \frac{1}{2}\partial_\sigma \mathbb{X}^A \left(H_{AB} \pm \eta_{AB}\right)\eta^{BC}\delta\eta_{CD}\left(\partial_+ \mathbb{X}^D + \partial_- \mathbb{X}^D\right) \\
&= \pm\partial_\pm \mathbb{X}^A \delta\eta_{AB}\left(\partial_+ \mathbb{X}^B + \partial_- \mathbb{X}^B\right).
\end{aligned}$$

Adding the terms, we get the full perturbation of the energy-momentum tensor:

$$\delta T_\pm = -\frac{1}{2}\partial_\pm \mathbb{X}^A \delta H_{AB}\partial_\pm \mathbb{X}^B - \frac{1}{2}\partial_+ \mathbb{X}^A\left(\delta H_{AB} \pm \delta\eta_{AB}\right)\partial_- \mathbb{X}^B \pm \partial_\pm \mathbb{X}^A \delta\eta_{AB}\left(\partial_+ \mathbb{X}^B + \partial_- \mathbb{X}^B\right). \tag{3.141}$$

In terms of the holomorphic and antiholomorphic components, we have:

$$\begin{aligned}
\delta T &= -\frac{1}{2}\partial \mathbb{X}^A \delta H_{AB}\partial \mathbb{X}^B - \frac{1}{2}\partial \mathbb{X}^A\left(\delta H_{AB} + \delta\eta_{AB}\right)\bar{\partial} \mathbb{X}^B + \partial \mathbb{X}^A \delta\eta_{AB}\left(\partial \mathbb{X}^B + \bar{\partial} \mathbb{X}^B\right) \\
&= -\frac{1}{2}\partial \mathbb{X}^A\left(\delta H_{AB} - 2\delta\eta_{AB}\right)\partial \mathbb{X}^B - \frac{1}{2}\partial \mathbb{X}^A\left(\delta H_{AB} - \delta\eta_{AB}\right)\bar{\partial} \mathbb{X}^B \\
&= \frac{1}{2}\partial \mathbb{X}^A \delta H^{++}_{AB}\partial \mathbb{X}^B - \frac{1}{2}\partial \mathbb{X}^A\left(\delta H^{+-}_{AB} - \delta\eta^{+-}_{AB}\right)\bar{\partial} \mathbb{X}^B,
\end{aligned}$$

$$\begin{aligned}
\delta \bar{T} &= -\frac{1}{2}\bar{\partial} \mathbb{X}^A \delta H_{AB}\bar{\partial} \mathbb{X}^B - \frac{1}{2}\partial \mathbb{X}^A\left(\delta H_{AB} - \delta\eta_{AB}\right)\bar{\partial} \mathbb{X}^B - \bar{\partial} \mathbb{X}^A \delta\eta_{AB}\left(\partial \mathbb{X}^B + \bar{\partial} \mathbb{X}^B\right) \\
&= -\frac{1}{2}\bar{\partial} \mathbb{X}^A\left(\delta H_{AB} + 2\delta\eta_{AB}\right)\bar{\partial} \mathbb{X}^B - \frac{1}{2}\partial \mathbb{X}^A\left(\delta H_{AB} + \delta\eta_{AB}\right)\bar{\partial} \mathbb{X}^B \\
&= \frac{1}{2}\bar{\partial} \mathbb{X}^A \delta H^{--}_{AB}\bar{\partial} \mathbb{X}^B - \frac{1}{2}\partial \mathbb{X}^A\left(\delta H^{+-}_{AB} + \delta\eta^{+-}_{AB}\right)\bar{\partial} \mathbb{X}^B.
\end{aligned}$$

The deformation of the worldsheet CFT obtained by perturbing the metrics is, according to our definitions in section 2.1, a chiral canonical deformation provided that both δT and $\delta\bar{T}$ are $(1,1)$ primary fields. In contrast with the achiral canonical deformation obtained for the usual string theory in section 2.6, here we have obtained a chiral one, with $\delta T \neq \delta\bar{T}$ in general, due to the chiral nature of metastring theory.

Observe that if we perturb only H and leave η unchanged then, in light of the relation $\delta H_{\pm\pm} = \pm\delta\eta_{\pm\pm}$, this chiral deformation automatically reduces to an achiral one where $\delta T = \delta\bar{T} = -\frac{1}{2}\partial\mathbb{X}H\bar{\partial}\mathbb{X}$, analogous to the case of standard string theory. This is further evidence that the P-metric η is a new feature of metastring theory.

Moreover, as we shall see later, if we perturb only the spacetime metric (and not the momentum-space metric), we completely reduce to the usual string theory case.

3.5 Derivation of the Linearized Metagravity Equations

3.5.1 A General Conformal Deformation

Let us deform the holomorphic and antiholomorphic components of the metastring energy-momentum tensor by arbitrary rank-2 tensors V and $\bar{V}$, respectively:

$$\delta T \equiv \partial_\sigma\mathbb{X}V\partial_\sigma\mathbb{X} = \partial\mathbb{X}V_{++}\partial\mathbb{X} + 2\partial\mathbb{X}V_{+-}\bar{\partial}\mathbb{X} + \bar{\partial}\mathbb{X}V_{--}\bar{\partial}\mathbb{X}, \tag{3.142}$$

$$\delta\bar{T} \equiv \partial_\sigma\mathbb{X}\bar{V}\partial_\sigma\mathbb{X} = \partial\mathbb{X}\bar{V}_{++}\partial\mathbb{X} + 2\partial\mathbb{X}\bar{V}_{+-}\bar{\partial}\mathbb{X} + \bar{\partial}\mathbb{X}\bar{V}_{--}\bar{\partial}\mathbb{X}. \tag{3.143}$$

Target space indices have been suppressed here. We do not assume a priori any symmetry properties of V and $\bar{V}$. We require that the deformation is a chiral conformal deformation, so that conformal symmetry is preserved. For this, it is enough to demand that each of the six individual terms in the expressions above is a $(1,1)$ primary field, by calculating their OPEs with T and $\bar{T}$. This calculation has been delegated to Appendix B.

As explained in the Appendix, it is enough to consider just δT and V, with the results applying equally to $\delta\bar{T}$ and $\bar{V}$. Let us convert the results, which are written in the Appendix in dense chiral notation, into a more readable form. We have three distinct cases:

- From $T_\pm\delta T_{\pm\pm}$ we get:

$$\Box_\pm V^{\pm\pm}_{AB} = 2V^{\pm\pm}_{AB}, \qquad H^{AC}_{\pm\pm}\partial_C V^{\pm\pm}_{(AB)} = 0, \qquad H^{AB}_{\pm\pm}V^{\pm\pm}_{AB} = 0. \tag{3.144}$$

- From $T_+\delta T_{+-}$ and $T_-\delta T_{+-}$ we get:

$$\Box_+ V^{+-}_{AB} = \Box_- V^{+-}_{AB} = 0, \qquad H^{AC}_{++}\partial_C V^{+-}_{AB} = H^{BC}_{--}\partial_C V^{+-}_{AB} = 0. \tag{3.145}$$

- From $T_\mp\delta T_{\pm\pm}$ we get:

$$\Box_\mp V^{\pm\pm}_{AB} = -2V^{\pm\pm}_{AB}. \tag{3.146}$$

Using

$$H_{\pm\pm} = \frac{1}{2}\left(H \pm \eta\right), \tag{3.147}$$

the chiral d'Alembertian becomes

$$\Box_\pm = \frac{1}{2}\left(H^{AB} \pm \eta^{AB}\right)\partial_A\partial_B \equiv \frac{1}{2}\left(\Box_H \pm \Box_\eta\right), \tag{3.148}$$

where we defined

$$\Box_H \equiv H^{AB}\partial_A\partial_B, \qquad \Box_\eta \equiv \eta^{AB}\partial_A\partial_B. \tag{3.149}$$

Therefore the various wave equations may be written as

$$\left(\Box_H + \Box_\eta\right)V^{++}_{AB} = +4V^{++}_{AB}, \qquad \left(\Box_H - \Box_\eta\right)V^{--}_{AB} = +4V^{--}_{AB}, \tag{3.150}$$

$$(\Box_H + \Box_\eta)\, V_{AB}^{+-} = (\Box_H - \Box_\eta)\, V_{AB}^{+-} = 0, \tag{3.151}$$

$$(\Box_H - \Box_\eta)\, V_{AB}^{++} = -4V_{AB}^{++}, \qquad (\Box_H + \Box_\eta)\, V_{AB}^{--} = -4V_{AB}^{--}. \tag{3.152}$$

By taking linear combinations of these equations we get

$$\Box_H V_{AB}^{++} = \Box_H V_{AB}^{--} = \Box_H V_{AB}^{+-} = 0, \tag{3.153}$$

$$\Box_\eta V_{AB}^{++} = +4V_{AB}^{++}, \qquad \Box_\eta V_{AB}^{--} = -4V_{AB}^{--}, \qquad \Box_\eta V_{AB}^{+-} = 0. \tag{3.154}$$

These six equations may be neatly summarized as follows:

$$\Box_H V_{AB}^{XY} = 0, \qquad \Box_\eta V_{AB}^{XY} = 2\left\{J, V_{AB}^{XY}\right\} = 2\,(X+Y)\, V_{AB}^{XY}, \qquad X, Y \in \{+,-\}. \tag{3.155}$$

The gauge conditions may be written using either H, η, or both, due to their projection properties. We will choose to write them using H:

$$H^{AC}\partial_C V_{(AB)}^{++} = H^{AC}\partial_C V_{(AB)}^{--} = 0, \tag{3.156}$$

$$H^{AC}\partial_C V_{AB}^{+-} = H^{BC}\partial_C V_{AB}^{+-} = 0. \tag{3.157}$$

Note that for $V^{\pm\pm}$ the conditions only apply to the symmetric part. Lastly, we have the tracelessness conditions, which we again choose to write in terms of H:

$$H^{AB}V_{AB}^{++} = H^{AB}V_{AB}^{--} = H^{AB}V_{AB}^{+-} = 0. \tag{3.158}$$

For V^{+-} we get this condition "for free", due to the off-diagonal projections of H and η vanishing.

3.5.2 The Linearized Metagravity Equations

Finally we are able to write down our main result, the linearized metagravity equations, using the results of the previous sections. In section 3.4.2 we found that the perturbations of T and $\bar{T}$ are:

$$\delta T = \frac{1}{2}\partial\mathbb{X}\delta H^{++}\partial\mathbb{X} - \frac{1}{2}\partial\mathbb{X}\left(\delta H^{+-} - \delta\eta^{+-}\right)\bar{\partial}\mathbb{X}, \tag{3.159}$$

$$\delta\bar{T} = \frac{1}{2}\bar{\partial}\mathbb{X}\delta H^{--}\bar{\partial}\mathbb{X} - \frac{1}{2}\partial\mathbb{X}\left(\delta H^{+-} + \delta\eta^{+-}\right)\bar{\partial}\mathbb{X}. \tag{3.160}$$

We thus have four independent deformations δT^{++}, δT^{+-}, $\delta\bar{T}^{--}$, $\delta\bar{T}^{+-}$ and four independent perturbations δH^{++}, δH^{--}, δH^{+-}, $\delta\eta^{+-}$. The equations of motion and gauge conditions for the perturbations are given using the results of section 3.5.1 by defining

$$\delta T \equiv \partial\mathbb{X}V^{++}\partial\mathbb{X} + \partial\mathbb{X}V^{+-}\bar{\partial}\mathbb{X}, \tag{3.161}$$

$$\delta\bar{T} \equiv \bar{\partial}\mathbb{X}\bar{V}^{--}\bar{\partial}\mathbb{X} + \partial\mathbb{X}\bar{V}^{+-}\bar{\partial}\mathbb{X}. \tag{3.162}$$

We identify

$$V^{++} \sim \delta H^{++}, \qquad \bar{V}^{--} \sim \delta H^{--}, \tag{3.163}$$

$$V^{+-} \sim \delta H^{+-} - \delta\eta^{+-}, \qquad \bar{V}^{+-} \sim \delta H^{+-} + \delta\eta^{+-}, \tag{3.164}$$

so that[14]

$$\delta H^{+-} \sim \frac{\bar{V}^{+-} + V^{+-}}{2}, \qquad \delta\eta^{+-} \sim \frac{\bar{V}^{+-} - V^{+-}}{2}. \tag{3.165}$$

Recalling the relation $\delta H^{\pm\pm} = \pm\delta\eta^{\pm\pm}$, we find the following equations of motion:

$$\Box_H \delta H_{AB}^{++} = \Box_H \delta H_{AB}^{--} = \Box_H \delta H_{AB}^{+-} = 0, \tag{3.166}$$

$$\Box_\eta \delta H_{AB}^{++} = +4\delta H_{AB}^{++}, \qquad \Box_\eta \delta H_{AB}^{--} = -4\delta H_{AB}^{--}, \qquad \Box_\eta \delta H_{AB}^{+-} = 0, \tag{3.167}$$

[14] In fact, the exact form of the linear combination doesn't matter, since all equations involving V^{+-} and $\bar{V}^{+-}$ have zero on the right-hand side anyway. It only matters that they are linearly independent.

$$\square_H \delta\eta^{++}_{AB} = \square_H \delta\eta^{--}_{AB} = \square_H \delta\eta^{+-}_{AB} = 0, \tag{3.168}$$

$$\square_\eta \delta\eta^{++}_{AB} = +4\delta\eta^{++}_{AB}, \qquad \square_\eta \delta\eta^{--}_{AB} = -4\delta\eta^{--}_{AB}, \qquad \square_\eta \delta\eta^{+-}_{AB} = 0. \tag{3.169}$$

Using the relation (valid for any symmetric V)

$$V_{AB} = V^{++}_{AB} + V^{--}_{AB} + 2V^{+-}_{(AB)}, \tag{3.170}$$

we may sum the equations above to obtain equations for the unprojected perturbations:

$$\square_H \delta H_{AB} = \square_H \delta\eta_{AB} = 0, \tag{3.171}$$

$$\square_\eta \delta H_{AB} = 4\left(\delta H^{++}_{AB} - \delta H^{--}_{AB}\right), \qquad \square_\eta \delta\eta_{AB} = 4\left(\delta\eta^{++}_{AB} - \delta\eta^{--}_{AB}\right). \tag{3.172}$$

The gauge conditions and tracelessness conditions are

$$H^{AC}\partial_C \delta H^{++}_{AB} = H^{AC}\partial_C \delta H^{--}_{AB} = H^{AC}\partial_C \delta H^{+-}_{AB} = 0, \tag{3.173}$$

$$H^{AC}\partial_C \delta\eta^{++}_{AB} = H^{AC}\partial_C \delta\eta^{--}_{AB} = H^{AC}\partial_C \delta\eta^{+-}_{AB} = 0, \tag{3.174}$$

$$H^{AB}\delta H^{++}_{AB} = H^{AB}\delta H^{--}_{AB} = H^{AB}\delta H^{+-}_{AB} = 0, \tag{3.175}$$

$$H^{AB}\delta\eta^{++}_{AB} = H^{AB}\delta\eta^{--}_{AB} = H^{AB}\delta\eta^{+-}_{AB} = 0. \tag{3.176}$$

These readily translate into conditions for the unprojected perturbations.

Let us summarize the complete set of equations for the unprojected perturbations of the two metrics H and η:

$$\square_H \delta H_{AB} = \square_H \delta\eta_{AB} = 0, \tag{3.177}$$

$$\square_\eta \delta H_{AB} = 4\left(\delta H^{++}_{AB} - \delta H^{--}_{AB}\right), \qquad \square_\eta \delta\eta_{AB} = 4\left(\delta\eta^{++}_{AB} - \delta\eta^{--}_{AB}\right), \tag{3.178}$$

$$H^{AC}\partial_C \delta H_{AB} = H^{AC}\partial_C \delta\eta_{AB} = H^{AB}\delta H_{AB} = H^{AB}\delta\eta_{AB} = 0. \tag{3.179}$$

The equations of motion with respect to $\square_H$, together with the gauge conditions and tracelessness conditions (which are also gauge conditions, of course), are similar to the linearized Einstein equations in the transverse traceless gauge, with the Q-metric H in place of the spacetime metric, and both δH and $\delta\eta$ in place of the perturbations of the metric. However, $\square_H$ also encodes possible dependence of the perturbations on momentum-space coordinates in addition to spacetime coordinates. The equations of motion with respect to $\square_\eta$ are completely new, and some of their implications will be analyzed promptly.

We call this set of equations *The Linearized Metagravity Equations*. Note that they are only given here in a particular gauge; the gauge-invariant linearized equations, as well as the full nonlinear equations, will be derived in future work.

3.6 Preliminary Analysis of the Metagravity Equations

3.6.1 Explicit Momentum Dependence of the Metageometry

It is instructive to write down explicitly the momentum dependence encoded in the linearized metagravity equations, that is, dependence on the momentum-space coordinates Y. We define the spacetime and momentum-space derivatives

$$\partial^X_\mu \equiv \frac{\partial}{\partial X^\mu}, \qquad \partial^\mu_Y \equiv \frac{\partial}{\partial Y_\mu}. \tag{3.180}$$

The explicit expressions for the d'Alembertians are then

$$\square_H = H^{CD}\partial_C\partial_D = \begin{pmatrix} \partial^X_\mu & \partial^\mu_Y \end{pmatrix} \begin{pmatrix} h^{\mu\nu} & 0 \\ 0 & h_{\mu\nu} \end{pmatrix} \begin{pmatrix} \partial^X_\nu \\ \partial^\nu_Y \end{pmatrix} = h^{\mu\nu}\partial^X_\mu\partial^X_\nu + h_{\mu\nu}\partial^\mu_Y\partial^\nu_Y, \tag{3.181}$$

$$\Box_\eta = \eta^{CD}\partial_C\partial_D = \begin{pmatrix} \partial^X_\mu & \partial^\mu_Y \end{pmatrix} \begin{pmatrix} 0 & \delta^\mu_\nu \\ \delta^\nu_\mu & 0 \end{pmatrix} \begin{pmatrix} \partial^X_\nu \\ \partial^\nu_Y \end{pmatrix} = 2\partial^X_\mu \partial^\mu_Y. \tag{3.182}$$

The linearized metagravity equations (in the given gauge) thus take the form

$$\left(h^{\mu\nu}\partial^X_\mu\partial^X_\nu + h_{\mu\nu}\partial^\mu_Y\partial^\nu_Y\right)\delta H_{AB} = 0, \qquad \left(h^{\mu\nu}\partial^X_\mu\partial^X_\nu + h_{\mu\nu}\partial^\mu_Y\partial^\nu_Y\right)\delta \eta_{AB} = 0, \tag{3.183}$$

$$\frac{1}{2}\partial^X_\mu\partial^\mu_Y\delta H_{AB} = \delta H^{++}_{AB} - \delta H^{--}_{AB}, \qquad \frac{1}{2}\partial^X_\mu\partial^\mu_Y\delta \eta_{AB} = \delta \eta^{++}_{AB} - \delta \eta^{--}_{AB}. \tag{3.184}$$

Schematically, we may write them as follows:

$$\left(\partial^2_X + \partial^2_Y\right)\delta H = 0, \qquad \left(\partial^2_X + \partial^2_Y\right)\delta \eta = 0, \tag{3.185}$$

$$\frac{1}{2}\partial_X\partial_Y\delta H = \delta H^{++} - \delta H^{--}, \qquad \frac{1}{2}\partial_X\partial_Y\delta\eta = \delta\eta^{++} - \delta\eta^{--}. \tag{3.186}$$

Observe that, since in general $\partial_X\partial_Y\delta H \neq 0$ and $\partial_X\partial_Y\delta\eta \neq 0$, the perturbations δH and $\delta\eta$ should depend on both X and Y, that is, both spacetime and momentum-space coordinates. Only in the special case where both right-hand sides vanish is it possible for the perturbations to depend only on spacetime coordinates. Since $\delta H^{\pm\pm} = \pm\delta\eta^{\pm\pm}$, both right-hand sides vanish if and only if all diagonal projections vanish, that is, $\delta H^{++} = \delta H^{--} = 0$ and similarly for $\delta\eta$. In fact, this is precisely the case of standard string theory, which we discuss next.

3.6.2 Reduction to Standard String Theory

As a consistency check, we should verify that the usual equations of motion are recovered when metastring theory is reduced back to the usual string theory. Let us recall the Polyakov action in the first-order formalism (3.13). With $\lambda \equiv \varepsilon \equiv 1$ for brevity, this action takes the form

$$S = \frac{1}{2\pi}\int \mathrm{d}^2\sigma \left(\partial_\tau X^\mu \partial_\sigma Y_\mu - \frac{1}{2}h_{\mu\nu}\partial_\sigma X^\mu\partial_\sigma X^\nu - \frac{1}{2}h^{\mu\nu}\partial_\sigma Y_\mu \partial_\sigma Y_\nu\right). \tag{3.187}$$

We then noted that

$$\frac{1}{2}h^{\mu\nu}\partial_\sigma Y_\mu\partial_\sigma Y_\nu + \frac{1}{2}h_{\mu\nu}\partial_\sigma X^\mu\partial_\sigma X^\nu = \frac{1}{2}\partial_\sigma \mathbb{X}^A H_{AB}\partial_\sigma\mathbb{X}^B, \tag{3.188}$$

$$\partial_\tau X^\mu\partial_\sigma Y_\mu = \frac{1}{2}\partial_\tau\mathbb{X}^A\left(\eta+\omega\right)_{AB}\partial_\sigma\mathbb{X}^B, \tag{3.189}$$

which allowed us to write the Lorentzian metastring action (3.25):

$$S = \frac{1}{4\pi}\int \mathrm{d}^2\sigma \left(\partial_\tau\mathbb{X}^A\left(\eta+\omega\right)_{AB}\partial_\sigma\mathbb{X}^B - \partial_\sigma\mathbb{X}^A H_{AB}\partial_\sigma\mathbb{X}^B\right). \tag{3.190}$$

Thus, if H, η and ω are all flat, this action readily reduces back to the flat spacetime Polyakov action. We shall therefore leave the metageometry unperturbed, and only perturb the spacetime metric $h_{\mu\nu}$; this is the only type of perturbation compatible with the usual Polyakov action. This only changes H, while η and ω remain untouched. Explicitly, if we perturb $h \mapsto h + \delta h$ then we get

$$H = \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \mapsto \begin{pmatrix} h+\delta h & 0 \\ 0 & h^{-1}-\delta h \end{pmatrix} \equiv H + \delta H, \tag{3.191}$$

where

$$\delta H \equiv \begin{pmatrix} \delta h & 0 \\ 0 & -\delta h \end{pmatrix}. \tag{3.192}$$

Let us project this perturbation onto the chiral eigenspaces. The chiral structure J has the explicit form

$$J = \begin{pmatrix} 0 & h^{-1} \\ h & 0 \end{pmatrix}, \tag{3.193}$$

so the projectors are

$$P_{\pm} = \frac{1}{2}(1 \pm J) = \frac{1}{2}\begin{pmatrix} 1 & \pm h^{-1} \\ \pm h & 1 \end{pmatrix}. \tag{3.194}$$

An explicit calculation yields:

$$\begin{aligned} \delta H^{XY} &= \frac{1}{4}\begin{pmatrix} \delta^{\beta}_{\alpha} & X h_{\alpha\beta} \\ X h^{\alpha\beta} & \delta^{\alpha}_{\beta} \end{pmatrix}\begin{pmatrix} \delta h_{\beta\gamma} & 0 \\ 0 & -\delta h^{\beta\gamma} \end{pmatrix}\begin{pmatrix} \delta^{\gamma}_{\delta} & Y h^{\gamma\delta} \\ Y h_{\gamma\delta} & \delta^{\delta}_{\gamma} \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} \delta h_{\alpha\gamma} & -X\delta h^{\gamma}_{\alpha} \\ X\delta h^{\alpha}_{\gamma} & -\delta h^{\alpha\gamma} \end{pmatrix}\begin{pmatrix} \delta^{\gamma}_{\delta} & Y h^{\gamma\delta} \\ Y h_{\gamma\delta} & \delta^{\delta}_{\gamma} \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} (1-XY)\,\delta h_{\alpha\delta} & (Y-X)\,\delta h^{\delta}_{\alpha} \\ (X-Y)\,\delta h^{\alpha}_{\delta} & (XY-1)\,\delta h^{\alpha\delta} \end{pmatrix}. \end{aligned}$$

Hence we see that

$$\delta H^{\pm\pm} = 0, \qquad \delta H^{\pm\mp} = \frac{1}{2}\begin{pmatrix} \delta h_{\alpha\delta} & \mp\delta h^{\delta}_{\alpha} \\ \pm\delta h^{\alpha}_{\delta} & -\delta h^{\alpha\delta} \end{pmatrix}. \tag{3.195}$$

Observe that only the off-diagonal terms are nonzero. This is in accordance with the analysis of the usual string case performed in section 2.6, where the deformation of the energy-momentum tensor only had a $\partial X\bar{\partial}X$ component. It is also compatible with the fact that $\delta\eta = 0$, and thus in particular $\delta\eta^{\pm\pm} = 0$, from which we immediately deduce that $\delta H^{\pm\pm} = \pm\delta\eta^{\pm\pm} = 0$.

We now invoke the linearized metagravity equations for δH:

$$\left(h^{\mu\nu}\partial^{X}_{\mu}\partial^{X}_{\nu} + h_{\mu\nu}\partial^{\mu}_{Y}\partial^{\nu}_{Y}\right)\delta H_{AB} = 0, \tag{3.196}$$

$$\partial^{X}_{\mu}\partial^{\mu}_{Y}\delta H_{AB} = 0. \tag{3.197}$$

Note that the right-hand side of the second equation vanishes since $\delta H^{++} - \delta H^{--} = 0$. Thus the second equation tells us that if δH depends on the spacetime coordinates X, it cannot also depend on momentum-space coordinates. The first equation then reduces to

$$\Box\delta h_{\mu\nu} = 0. \tag{3.198}$$

This is the same result we obtained before for the Polyakov string. Similarly, for the gauge conditions we use

$$H^{AC}\partial_C = \begin{pmatrix} h^{\alpha\beta} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix}\begin{pmatrix} \partial^{X}_{\beta} \\ \partial^{\beta}_{Y} \end{pmatrix} = \begin{pmatrix} h^{\alpha\beta}\partial^{X}_{\beta} \\ h_{\alpha\beta}\partial^{\beta}_{Y} \end{pmatrix}, \tag{3.199}$$

$$\eta^{AC}\partial_C = \begin{pmatrix} 0 & \delta^{\alpha}_{\beta} \\ \delta^{\beta}_{\alpha} & 0 \end{pmatrix}\begin{pmatrix} \partial^{X}_{\beta} \\ \partial^{\beta}_{Y} \end{pmatrix} = \begin{pmatrix} \partial^{\alpha}_{Y} \\ \partial^{X}_{\alpha} \end{pmatrix}, \tag{3.200}$$

to get

$$\begin{aligned} 0 = H^{AC}\partial_C\delta H_{AB} &= \frac{1}{2}\begin{pmatrix} h^{\alpha\beta}\partial^{X}_{\beta} & h_{\alpha\beta}\partial^{\beta}_{Y} \end{pmatrix}\begin{pmatrix} \delta h_{\alpha\delta} & -\delta h^{\delta}_{\alpha} \\ \delta h^{\alpha}_{\delta} & -\delta h^{\alpha\delta} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \partial^{X}_{\alpha}\delta h^{\alpha}_{\delta} & -\partial^{X}_{\alpha}\delta h^{\alpha\delta} \end{pmatrix}, \end{aligned}$$

which gives the familiar gauge condition

$$\partial_{\mu}\delta h^{\mu\nu} = 0. \tag{3.201}$$

(If we had used η^{AC} in the gauge condition instead, we would have obtained the same result, as indeed we must.) Finally, we have the tracelessness condition

$$\begin{aligned} 0 = H^{AB}\delta H_{AB} &= \begin{pmatrix} h^{\alpha\beta} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix}\begin{pmatrix} \delta h_{\alpha\delta} & -\delta h^{\delta}_{\alpha} \\ \delta h^{\alpha}_{\delta} & -\delta h^{\alpha\delta} \end{pmatrix} \\ &= \left.\begin{pmatrix} h^{\alpha\beta}\delta h_{\alpha\delta} & -h^{\alpha\beta}\delta h^{\delta}_{\alpha} \\ h_{\alpha\beta}\delta h^{\alpha}_{\delta} & -h_{\alpha\beta}\delta h^{\alpha\delta} \end{pmatrix}\right|_{\delta\to\beta}, \end{aligned}$$

which gives simply

$$h^{\alpha\beta}\delta h_{\alpha\beta} = 0. \tag{3.202}$$

We thus see that the linearized metagravity equations automatically reduce to the usual linearized Einstein equations obtained for the Polyakov string if we perturb only the spacetime metric.

4 Conclusions

4.1 Summary of Findings

In this work we took a particularly attractive path towards our destination: the path of mathematical consistency. Inspired by the well-known result from string theory, where the theory is simply not mathematically consistent unless the background fields obey Einstein's equations, we set out to apply the same treatment to metastring theory. We have found that in order to preserve the conformal symmetry of the worldsheet, the metastring can't help but impose consistency conditions on the geometry of phase space.

The resulting equations, which we have optimistically dubbed the linearized metagravity equations, govern the dynamics of small perturbations about a flat phase space metageometry. The equations of motion for perturbations of the Q-metric δH and perturbations of the P-metric $\delta\eta$ may be written schematically as

$$\left(\partial_X^2 + \partial_Y^2\right)\delta H = 0, \qquad \left(\partial_X^2 + \partial_Y^2\right)\delta\eta = 0, \tag{4.1}$$

$$\frac{1}{2}\partial_X\partial_Y\delta H = \delta H^{++} - \delta H^{--}, \qquad \frac{1}{2}\partial_X\partial_Y\delta\eta = \delta\eta^{++} - \delta\eta^{--}. \tag{4.2}$$

These equations are given to us in a particular gauge, and a generalization to gauge-invariant equations is necessary. In addition they are, of course, merely linearized equations for perturbations over a fixed background metageometry, and their full nonlinear form, governing the dynamics of the background itself nonperturbatively, remains to be determined.

However, even in this preliminary form, they already show that both metrics, in general, should have energy-momentum as well as spacetime dependence. They also show that η, which tells us what spacetime is, can be dynamical. It thus seems reasonable to hope that the full nonlinear formulation of metagravity, which should follow naturally from further investigations into metastring theory and its metageometry, will provide a concrete realization of relative locality, where momentum space is curved, and the geometry of spacetime – as well as spacetime itself – are energy and momentum dependent.

4.2 The Road Ahead

The observant reader will note that the central chapter of this work is entitled "Conformal Deformations in Metastring Theory: Part I", yet Part II is nowhere to be found. This is just the very beginning of the study of conformal deformations in metastring theory; the next parts have not been written yet. In the future, unbounded by the time and size constraints of this essay, we wish to proceed as follows:

1. Find the linearized equations of motion for perturbations of the symplectic form ω, if possible.
2. Find the full symmetries obeyed by the perturbations δH, $\delta\eta$ and $\delta\omega$, using additional conformal deformation techniques not covered in this work.
3. Find the gauge-invariant linearized equations of motion for H, η and ω; these are the analogues of the linearized Einstein equations for metagravity.
4. Upgrade the linearized equations of motion to full, nonlinear equations of motion, analogous to the Einstein equations $G_{\mu\nu} = T_{\mu\nu}$.
5. Find an appropriate low-energy effective action for the metastring background, analogous to the Einstein-Hilbert action $\int \sqrt{-g}\, R$.
6. Generalize the analysis outlined above for general backgrounds, including the B-field and the dilaton.

Once the full nonlinear equations and action are found, metagravity is expected to be a generalization of Einstein gravity, which reduces to it in the appropriate limit, and obeys the principle of relative locality. This generalized theory may be studied in a variety of ways, both from the fundamental (meta)stringy perspective and as an effective theory with no explicit mention of (meta)strings.

Along the way, we have also uncovered some interesting research avenues which might have been overlooked when conformal deformations of string theory were first studied in the 1990s, and these will also be addressed in potential future work.

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A Calculation of the Energy-Momentum Tensor in Standard String Theory

We calculate the energy-momentum tensor using

$$T_{ab} \equiv -\frac{4\pi}{\sqrt{\gamma}}\frac{\delta S}{\delta\gamma^{ab}}. \tag{A.1}$$

When varying the determinant of the metric, we use

$$\delta\gamma \equiv \delta\left(\det\gamma_{ab}\right) = \delta\left(\exp\left(\operatorname{tr}\log\gamma_{ab}\right)\right) = \gamma\gamma^{ab}\delta\gamma_{ab}, \tag{A.2}$$

and

$$0 = \delta\left(\gamma^{ab}\gamma_{ab}\right) = \gamma^{ab}\delta\gamma_{ab} + \gamma_{ab}\delta\gamma^{ab}, \tag{A.3}$$

to get

$$\delta\gamma = -\gamma\gamma_{ab}\delta\gamma^{ab}. \tag{A.4}$$

This then gives

$$\delta\sqrt{\gamma} = \frac{1}{2\sqrt{\gamma}}\delta\gamma = -\frac{1}{2}\sqrt{\gamma}\gamma_{ab}\delta\gamma^{ab}. \tag{A.5}$$

Thus, varying the Euclidean action

$$S = \frac{1}{8\pi}\int \mathrm{d}^2\sigma\,\sqrt{\gamma}\,\gamma^{ab}\partial_a X^\mu \partial_b X^\nu h_{\mu\nu}, \tag{A.6}$$

we obtain

$$\delta S = \frac{1}{8\pi}\int \mathrm{d}^2\sigma\,\sqrt{\gamma}\,h_{\mu\nu}\left(\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\gamma_{ab}\partial_c X^\mu \partial^c X^\nu\right)\delta\gamma^{ab}, \tag{A.7}$$

so

$$T_{ab} = -\frac{1}{2}h_{\mu\nu}\left(\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\gamma_{ab}\partial_c X^\mu \partial^c X^\nu\right). \tag{A.8}$$

Note that this definition of T_{ab} automatically produces a symmetric tensor, as γ_{ab} is symmetric. As will be seen later, in metastring theory we cannot generally assume that the energy-momentum tensor is symmetric, so an alternative definition in terms of frame fields is in order. We note that such a definition is also needed, for example, when defining spinor fields on curved manifolds [11, App. 2.C]. We define the frame fields ∂_l and co-frame fields e^l by

$$e^l \equiv e^l{}_a \, \mathrm{d}\sigma^a, \qquad \partial_l \equiv e_l{}^a \partial_a, \qquad g_{lm} e^l{}_a e^m_{\ b} = \gamma_{ab}, \qquad \gamma_{ab} e_l{}^a e_m{}^b = g_{lm}, \qquad \partial_l e^m = \delta_l^m. \tag{A.9}$$

Here a, b are Lorentzian worldsheet indices and γ_{ab} is the Lorentzian worldsheet metric, while l, m are internal indices (taking the values $0, 1$) and g_{lm} is the internal Minkowski metric. Note that in our notation the internal index is always to the left of the worldsheet index. We may contract with the frame fields to convert indices on any tensor between worldsheet and internal indices, for example:

$$A^l \equiv e^l{}_a A^a, \qquad A^a \equiv e_l{}^a A^l. \tag{A.10}$$

Since $\gamma_{ab} = g_{lm} e^l{}_a e^m_{\ b}$, the metric may be written as

$$\mathrm{d}s^2 \equiv \gamma_{ab} \, \mathrm{d}\sigma^a \otimes \mathrm{d}\sigma^b = g_{lm} \left(e^l{}_a \, \mathrm{d}\sigma^a \right) \otimes \left(e^m_{\ b} \, \mathrm{d}\sigma^b \right) = -e^0 \otimes e^0 + e^1 \otimes e^1. \tag{A.11}$$

The Lorentzian Polyakov action thus becomes

$$S[X] = -\frac{1}{8\pi} \int \mathrm{d}^2\sigma \, \det(e) \; g^{lm} \partial_l X^\mu \partial_m X^\nu h_{\mu\nu}, \tag{A.12}$$

where we used

$$\sqrt{-\gamma} \equiv \sqrt{-\det \gamma_{ab}} = \sqrt{-\det \left(g_{lm} e^l{}_a e^m_{\ b} \right)} = \det e^l{}_a \equiv \det(e), \tag{A.13}$$

since $\det g = -1$. We then define the energy-momentum tensor with one worldsheet index and one internal index as:

$$T^l{}_a \equiv \frac{2\pi}{\det(e)} \frac{\delta S}{\delta e_l{}^a}. \tag{A.14}$$

Note the factor of $+2\pi$ compared to -4π in the definition using variation of the metric. We may further contract the internal index to obtain an expression with two worldsheet indices:

$$T_{ab} = \gamma_{ac} e_l{}^c T^l{}_b = \frac{2\pi \gamma_{ac} e_l{}^c}{\det(e)} \frac{\delta S}{\delta e_l{}^b}. \tag{A.15}$$

To calculate this, first we note that

$$\delta \det(e) = \delta \left(\exp \left(\operatorname{tr} \log e^l{}_a \right) \right) = \det(e) \, e_l{}^a \delta e^l{}_a \tag{A.16}$$

and

$$0 = \delta \left(e_l{}^a e^l{}_a \right) = e_l{}^a \delta e^l{}_a + e^l{}_a \delta e_l{}^a, \tag{A.17}$$

so

$$\delta \det(e) = -\det(e) \, e^l{}_a \delta e_l{}^a. \tag{A.18}$$

Also

$$\delta \partial_l = \delta e_l{}^a \partial_a = \delta e_l{}^a e^m_{\ a} \partial_m. \tag{A.19}$$

Thus we get

$$\delta S = -\frac{1}{8\pi} \int \mathrm{d}^2\sigma \, \det(e) \, h_{\mu\nu} g^{lm} \left(-e^l{}_b \partial_l X^\mu \partial_m X^\nu + 2 e^n_{\ b} \partial_n X^\mu \partial_m X^\nu \right) \delta e_l{}^b, \tag{A.20}$$

and so

$$\begin{aligned} T_{ab} &= -\frac{1}{2} \gamma_{ac} e_l{}^c h_{\mu\nu} g^{lm} \left(e^n_{\ b} \partial_n X^\mu \partial_m X^\nu - \frac{1}{2} e^l{}_b \partial_l X^\mu \partial_m X^\nu \right) \\ &= -\frac{1}{2} h_{\mu\nu} \left(\partial_a X^\mu \partial_b X^\nu - \frac{1}{2} \gamma_{ab} \partial_c X^\mu \partial^c X^\nu \right), \end{aligned}$$

which is the same as the result obtained using variation of the metric. Therefore, our choice of normalization was correct.

B Calculation of the $T_X \delta T_{YZ}$ OPE

First, note that the OPE calculation merely tells us what conditions must be satisfied in order for the perturbation to be a $(1,1)$ primary field; it is blind to whether the perturbation being considered is δT or $\delta\bar{T}$. Therefore we shall perform the calculation only for δT and V. The results will equally apply to $\delta\bar{T}$ and $\bar{V}$.

We calculate the OPEs with respect to each of the three projections V_{XY} individually. To facilitate this calculation, we first Fourier-transform V_{XY}:

$$V_{XY}(\mathbb{X}) \equiv \int \frac{\mathrm{d}^{2D}\mathbb{P}}{(2\pi)^{2D}} \Pi_{XY}(\mathbb{P})\, \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}}, \tag{B.1}$$

where $\mathbb{P}$ is a phase-space "momentum vector" and Π is a polarization tensor. Note that $\mathbb{P}$ and $\Pi_{XY}(\mathbb{P})$ are constant on the worldsheet, i.e., not functions of $z, \bar{z}$. The notation $\mathbb{P}\cdot\mathbb{X}$ is shorthand for $\mathbb{P}_A \mathbb{X}^A$. We deform our energy-momentum tensor by a single plane wave of momentum $\mathbb{P}$:

$$\delta T_{XY} \equiv \Pi^{XY}_{AB} \partial_X \mathbb{X}^A \partial_Y \mathbb{X}^B\, \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}}. \tag{B.2}$$

Our chiral notation allows us to calculate all six OPEs of $T_+ \equiv T$ and $T_- \equiv \bar{T}$ with the projections δT_{XY} as a single, general OPE, and then plug in $X, Y, Z \in \{+, -\}$:

$$T_X(X)\,\delta T_{YZ}(0,0) = -\frac{1}{2} H^{XX}_{AB} \Pi^{YZ}_{CD} : \partial_X \mathbb{X}^A(X)\, \partial_X \mathbb{X}^B(X) :: \partial_Y \mathbb{X}^C(0)\, \partial_Z \mathbb{X}^D(0)\, \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : . \tag{B.3}$$

First, we sum on all possible contractions. Note that we can contract with $\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}}$ twice. Since H_{AB} is symmetric, replacing $A \leftrightarrow B$ in a contraction will result in an equivalent contraction, but we're not assuming anything about the symmetry of Π_{CD}.

$$\begin{aligned}
&T_X(X)\,\delta T_{YZ}(0,0) = \\
&\quad -\frac{1}{2} H^{XX}_{AB} \Pi^{YZ}_{CD} : \partial_X \mathbb{X}^B(X)\, \partial_Y \mathbb{X}^C(0)\, \partial_Z \mathbb{X}^D(0) : \left\langle \partial_X \mathbb{X}^A(X)\, \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} \right\rangle + (A \leftrightarrow B) + \\
&\quad -\frac{1}{2} H^{XX}_{AB} \Pi^{YZ}_{CD} : \partial_X \mathbb{X}^B(X)\, \partial_Z \mathbb{X}^D(0)\, \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : \left\langle \partial_X \mathbb{X}^A(X)\, \partial_Y \mathbb{X}^C(0) \right\rangle + (A \leftrightarrow B) + \\
&\quad -\frac{1}{2} H^{XX}_{AB} \Pi^{YZ}_{CD} : \partial_X \mathbb{X}^B(X)\, \partial_Y \mathbb{X}^C(0)\, \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : \left\langle \partial_X \mathbb{X}^A(X)\, \partial_Z \mathbb{X}^D(0) \right\rangle + (A \leftrightarrow B) + \\
&\quad -\frac{1}{2} H^{XX}_{AB} \Pi^{YZ}_{CD} : \partial_Y \mathbb{X}^C(0)\, \partial_Z \mathbb{X}^D(0) : \left\langle \partial_X \mathbb{X}^A(X)\, \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} \right\rangle \left\langle \partial_X \mathbb{X}^B(X)\, \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} \right\rangle + \\
&\quad -\frac{1}{2} H^{XX}_{AB} \Pi^{YZ}_{CD} : \partial_Z \mathbb{X}^D(0) : \left\langle \partial_X \mathbb{X}^A(X)\, \partial_Y \mathbb{X}^C(0) \right\rangle \left\langle \partial_X \mathbb{X}^B(X)\, \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} \right\rangle + (A \leftrightarrow B) + \\
&\quad -\frac{1}{2} H^{XX}_{AB} \Pi^{YZ}_{CD} : \partial_Y \mathbb{X}^C(0) : \left\langle \partial_X \mathbb{X}^A(X)\, \partial_Z \mathbb{X}^D(0) \right\rangle \left\langle \partial_X \mathbb{X}^B(X)\, \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} \right\rangle + (A \leftrightarrow B) + \\
&\quad -\frac{1}{2} H^{XX}_{AB} \Pi^{YZ}_{CD} : \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : \left\langle \partial_X \mathbb{X}^A(X)\, \partial_Y \mathbb{X}^C(0) \right\rangle \left\langle \partial_X \mathbb{X}^B(Y)\, \partial_Z \mathbb{X}^D(0) \right\rangle + (A \leftrightarrow B).
\end{aligned}$$

Writing the contractions explicitly, we get

$$\begin{aligned}
T_X(X)\,\delta T_{YZ}(0,0) = {} & \frac{1}{X}\, \mathrm{i}\,\mathbb{P}_E H^{AE}_{XX} H^{XX}_{AB} \Pi^{YZ}_{CD} : \partial_X \mathbb{X}^B(X)\, \partial_Y \mathbb{X}^C(0)\, \partial_Z \mathbb{X}^D(0)\, \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& + \frac{1}{X^2} H^{AC}_{XY} H^{XX}_{AB} \Pi^{YZ}_{CD} : \partial_X \mathbb{X}^B(X)\, \partial_Z \mathbb{X}^D(0)\, \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& + \frac{1}{X^2} H^{AD}_{XZ} H^{XX}_{AB} \Pi^{YZ}_{CD} : \partial_X \mathbb{X}^B(X)\, \partial_Y \mathbb{X}^C(0)\, \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& + \frac{1}{X^2} \frac{1}{2} \mathbb{P}_E \mathbb{P}_F H^{AE}_{XX} H^{BF}_{XX} H^{XX}_{AB} \Pi^{YZ}_{CD} : \partial_Y \mathbb{X}^C(0)\, \partial_Z \mathbb{X}^D(0)\, \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& - \frac{1}{X^3}\, \mathrm{i}\,\mathbb{P}_E H^{BE}_{XX} H^{AC}_{XY} H^{XX}_{AB} \Pi^{YZ}_{CD} : \partial_Z \mathbb{X}^D(0)\, \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& - \frac{1}{X^3}\, \mathrm{i}\,\mathbb{P}_E H^{BE}_{XX} H^{AD}_{XZ} H^{XX}_{AB} \Pi^{YZ}_{CD} : \partial_Y \mathbb{X}^C(0)\, \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& - \frac{1}{X^4} H^{AC}_{XY} H^{BD}_{XZ} H^{XX}_{AB} \Pi^{YZ}_{CD} : \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : .
\end{aligned}$$

Next, we simplify the H's:

$$
\begin{aligned}
T_X(X)\,\delta T_{YZ}(0,0) = {} & \frac{1}{X}\,\mathrm{i}\,\mathbb{P}^X_B \Pi^{YZ}_{CD} : \partial_X \mathbb{X}^B(X)\,\partial_Y \mathbb{X}^C(0)\,\partial_Z \mathbb{X}^D(0)\,\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& + \frac{1}{X^2}\delta_{XY}\Pi^{YZ}_{CD} : \partial_X \mathbb{X}^C(X)\,\partial_Z \mathbb{X}^D(0)\,\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& + \frac{1}{X^2}\delta_{XZ}\Pi^{YZ}_{CD} : \partial_X \mathbb{X}^D(X)\,\partial_Y \mathbb{X}^C(0)\,\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& + \frac{1}{X^2}\frac{1}{2}\,|\mathbb{P}|^2_X\,\Pi^{YZ}_{CD} : \partial_Y \mathbb{X}^C(0)\,\partial_Z \mathbb{X}^D(0)\,\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& - \frac{1}{X^3}\delta_{XY}\,\mathrm{i}\,\mathbb{P}^X_E H^{CE}_{XX}\Pi^{YZ}_{CD} : \partial_Z \mathbb{X}^D(0)\,\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& - \frac{1}{X^3}\delta_{XZ}\,\mathrm{i}\,\mathbb{P}^X_E H^{DE}_{XX}\Pi^{YZ}_{CD} : \partial_Y \mathbb{X}^C(0)\,\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& - \frac{1}{X^4}\delta_{XY}\delta_{YZ}H^{CD}_{XX}\Pi^{YZ}_{CD} : \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : .
\end{aligned}
$$

Here we defined the projections of a phase-space covector:

$$
\mathbb{P}_X \equiv P_X^{\mathrm{T}}\mathbb{P}, \qquad (\mathbb{P}_X)_A \equiv (P_X)^B{}_A\,\mathbb{P}_B, \tag{B.4}
$$

as well as its projected norm-squared:

$$
|\mathbb{P}|^2_X \equiv \mathbb{P}H^{-1}_{XX}\mathbb{P} = \mathbb{P}^X_A H^{AB}_{XX}\mathbb{P}^X_B. \tag{B.5}
$$

Next, we expand $\partial_X \mathbb{X}(X)$ around $X=0$:

$$
\partial_X \mathbb{X}(X) \approx \partial_X \mathbb{X}(0) + X\partial_X\partial_X \mathbb{X}(0). \tag{B.6}
$$

This gives, discarding terms regular as $X \to 0$:

$$
\begin{aligned}
T_X(X)\,\delta T_{YZ}(0,0) = {} & \frac{1}{X}\,\mathrm{i}\,\mathbb{P}^X_B \Pi^{YZ}_{CD} : \partial_X \mathbb{X}^B(0)\,\partial_Y \mathbb{X}^C(0)\,\partial_Z \mathbb{X}^D(0)\,\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& + \frac{1}{X}\delta_{XY}\Pi^{YZ}_{CD} : \partial_X\partial_Y \mathbb{X}^C(0)\,\partial_Z \mathbb{X}^D(0)\,\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& + \frac{1}{X}\delta_{XZ}\Pi^{YZ}_{CD} : \partial_Y \mathbb{X}^C(0)\,\partial_X\partial_Z \mathbb{X}^D(0)\,\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : \\
& + \frac{1}{X^2}\delta_{XY}\Pi^{YZ}_{CD} : \partial_Y \mathbb{X}^C(0)\,\partial_Z \mathbb{X}^D(0)\,\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& + \frac{1}{X^2}\delta_{XZ}\Pi^{YZ}_{CD} : \partial_Y \mathbb{X}^C(0)\,\partial_Z \mathbb{X}^D(0)\,\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& + \frac{1}{X^2}\frac{1}{2}\,|\mathbb{P}|^2_X\,\Pi^{YZ}_{CD} : \partial_Y \mathbb{X}^C(0)\,\partial_Z \mathbb{X}^D(0)\,\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& - \frac{1}{X^3}\delta_{XY}\,\mathrm{i}\,\mathbb{P}^X_E H^{CE}_{XX}\Pi^{YZ}_{CD} : \partial_Z \mathbb{X}^D(0)\,\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& - \frac{1}{X^3}\delta_{XZ}\,\mathrm{i}\,\mathbb{P}^X_E H^{DE}_{XX}\Pi^{YZ}_{CD} : \partial_Y \mathbb{X}^C(0)\,\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& - \frac{1}{X^4}\delta_{XY}\delta_{YZ}H^{CD}_{XX}\Pi^{YZ}_{CD} : \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : .
\end{aligned}
$$

We identify the coefficient of $1/X$ as a derivative and consolidate the coefficients of $1/X^2$ and $1/X^3$:

$$
\begin{aligned}
T_X(X)\,\delta T_{YZ}(0,0) = {} & \frac{1}{X}\partial_X\left(\Pi^{YZ}_{CD} : \partial_Y \mathbb{X}^C(0)\,\partial_Z \mathbb{X}^D(0)\,\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} :\right) \\
& + \frac{1}{X^2}\left(\delta_{XY} + \delta_{XZ} + \frac{1}{2}\,|\mathbb{P}|^2_X\right)\Pi^{YZ}_{CD} : \partial_Y \mathbb{X}^C(0)\,\partial_Z \mathbb{X}^D(0)\,\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& - \frac{1}{X^3}\,\mathrm{i}\,H^{DE}_{XX}\mathbb{P}^X_E : \left(\Pi^{YZ}_{DC}\delta_{XY}\partial_Z \mathbb{X}^C(0) + \Pi^{YZ}_{CD}\delta_{XZ}\partial_Y \mathbb{X}^C(0)\right)\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : + \\
& - \frac{1}{X^4}\delta_{XY}\delta_{YZ}H^{CD}_{XX}\Pi^{YZ}_{CD} : \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : .
\end{aligned}
$$

Inserting the definition of δT_{YZ}, we finally obtain the OPE

$$T_X(X)\,\delta T_{YZ}(0,0) \sim -\frac{1}{X^4}\delta_{XY}\delta_{YZ}H^{AB}_{XX}\Pi^{YZ}_{AB} : \mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : +$$
$$-\frac{1}{X^3}\,\mathrm{i}\,H^{AC}_{XX}\mathbb{P}^X_C : \left(\Pi^{YZ}_{AB}\delta_{XY}\partial_Z\mathbb{X}^B(0) + \Pi^{YZ}_{BA}\delta_{XZ}\partial_Y\mathbb{X}^B(0)\right)\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}(0,0)} : +$$
$$+\frac{1}{X^2}\left(\delta_{XY}+\delta_{XZ}+\frac{1}{2}\left|\mathbb{P}\right|^2_X\right)\delta T_{YZ}(0,0) + \frac{1}{X}\partial_X\delta T_{YZ}(0,0)\,.$$

Demanding that each of the projections δT_{XY} will be a $(1,1)$ primary field yields the following conditions:

$$T_X(X)\,\delta T_{YZ}(0,0) \sim \frac{1}{X^2}\delta T_{YZ}(0,0) + \frac{1}{X}\partial_X\delta T_{YZ}(0,0)\,, \qquad X,Y,Z \in \{+,-\}\,. \tag{B.7}$$

These conditions are satisfied for all X, Y, Z if:

$$\left|\mathbb{P}\right|^2_X = 2\left(1-\delta_{XY}-\delta_{XZ}\right), \qquad \delta_{XY}\delta_{YZ}H^{AB}_{XX}\Pi^{YZ}_{AB} = 0, \tag{B.8}$$

$$H^{AC}_{XX}\mathbb{P}^X_C : \left(\Pi^{YZ}_{AB}\delta_{XY}\partial_Z\mathbb{X}^B + \Pi^{YZ}_{BA}\delta_{XZ}\partial_Y\mathbb{X}^B\right)\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}} := 0. \tag{B.9}$$

Alternatively, we may write them in terms of a single plane wave $V^{YZ}_{AB} = \Pi^{YZ}_{AB}\,\mathrm{e}^{\mathrm{i}\,\mathbb{P}\cdot\mathbb{X}}$, using the target space derivatives ∂_A and chiral target space d'Alembertian $\Box_X$:

$$\partial_A \equiv \frac{\partial}{\partial\mathbb{X}^B}, \qquad \Box_X \equiv H^{AB}_{XX}\partial_A\partial_B, \tag{B.10}$$

such that

$$\partial_C V^{YZ}_{AB} = \mathrm{i}\,\mathbb{P}_C V^{YZ}_{AB}, \qquad \Box_X V^{YZ}_{AB} = -\left|\mathbb{P}\right|^2_X V^{YZ}_{AB}. \tag{B.11}$$

We get

$$\Box_X V^{YZ}_{AB} = 2\left(\delta_{XY}+\delta_{XZ}-1\right)V^{YZ}_{AB}, \qquad \delta_{XY}\delta_{YZ}H^{AB}_{XX}V^{YZ}_{AB} = 0, \tag{B.12}$$

$$H^{AC}_{XX}\left(\delta_{XY}\partial_Z\mathbb{X}^B\partial_C V^{YZ}_{AB} + \delta_{XZ}\partial_Y\mathbb{X}^B\partial_C V^{YZ}_{BA}\right) = 0. \tag{B.13}$$

The meaning of these conditions is clarified in the main text.

References

[1] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, and L. Smolin, "The principle of relative locality," *Phys.Rev.* **D84** (2011) 084010, `arXiv:1101.0931 [hep-th]`.

[2] L. Freidel, R. G. Leigh, and D. Minic, "Metastring Theory and Modular Space-time," `arXiv:1502.08005 [hep-th]`.

[3] L. Freidel, R. G. Leigh, and D. Minic, "Born Reciprocity in String Theory and the Nature of Spacetime," *Phys.Lett.* **B730** (2014) 302–306, `arXiv:1307.7080`.

[4] L. Freidel, R. G. Leigh, and D. Minic, "Quantum Gravity, Dynamical Phase Space and String Theory," *Int.J.Mod.Phys.* **D23** no. 12, (2014) 1442006, `arXiv:1405.3949 [hep-th]`.

[5] M. Evans and B. A. Ovrut, "Deformations of Conformal Field Theories and Symmetries of the String," *Phys.Rev.* **D41** (1990) 3149.

[6] M. Evans and I. Giannakis, "Gauge covariant deformations, symmetries and free parameters of string theory," *Phys.Rev.* **D44** (1991) 2467–2479.

[7] A. Sen, "On the Background Independence of String Field Theory," *Nucl.Phys.* **B345** (1990) 551–583.

[8] M. Campbell, P. C. Nelson, and E. Wong, "Stress tensor perturbations in conformal field theory," *Int.J.Mod.Phys.* **A6** (1991) 4909–4924.

[9] J. Polchinski, *String Theory: Volume 1, An Introduction to the Bosonic String.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1998.

[10] G. Aldazabal, D. Marques, and C. Nunez, "Double Field Theory: A Pedagogical Review," *Class.Quant.Grav.* **30** (2013) 163001, `arXiv:1305.1907 [hep-th]`.

[11] P. D. Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory.* Graduate Texts in Contemporary Physics. Springer, 1997.